DIRECTORATE OF EDUCATION
Govt. of NCT, Delhi

SUPPORT MATERIAL
2025-2026

Class : XII

MATHEMATICS

Under the Guidance of

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Secretary (Education)

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पांडुरंग के. पोले, भा.प्र.से
सचिव (शिक्षा)

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SECRETARY (Education)

सत्यमेव जयते

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D.O. NO. : DE.5/228/Exam/Message/S.M/248/249
Date : 07/11/2025

MESSAGE

The Directorate of Education remains steadfast in its vision to achieve excellence in the academic domain and its commitment to develop meaningful, engaging, and child-friendly learning content.

Each year, the Directorate carefully reviews and updates the Support Material to ensure alignment with the latest CBSE guidelines and emerging academic developments.

The Support Material provides comprehensive academic support through well-structured practice questions and exercises that strengthen conceptual understanding and exam readiness and aims to nurture students' critical thinking, analytical abilities, and problem-solving skills. Through such sustained efforts, the Directorate of Education continues to guide students towards academic excellence and holistic growth.

This Support Material is intended to bridge classroom learning and examination preparation, enabling students to consolidate knowledge through systematic practice. It has been thoughtfully designed for students, with the belief that its effective use will strengthen their understanding and support them in achieving their learning goals with confidence.

I appreciate the dedication and collaborative effort of all those involved in the development of this material and extend my best wishes to all students —may this Support Material serve as an essential academic aid, enhancing students' confidence and preparedness for examinations.

Best Wishes.

(Pandurang K. Pole)

VEDITHA REDDY, IAS
Director, Education & Sports

सत्यमेव जयते

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DE-5/228/Exam/Message/S-M/2018/402
dated - 09/05/25

MESSAGE

Education is the cornerstone of a progressive society, and providing students with the right learning resources is essential for their academic and personal growth. Keeping this in mind, the Directorate of Education, GNCT of Delhi, develops comprehensive Support Material every year for various subjects of Classes IX to XII.

The support material serves as an additional study resource to supplement textbooks by offering clear and easy-to-understand explanation of complex topics. Our dedicated team of expert faculty members has meticulously reviewed and updated this material, aligning it with the latest CBSE syllabus, question paper pattern and assessment guidelines. Our effort is to simplify difficult concepts and make them more accessible to students, helping them save time and effort with ready references for effective preparation.

As Ruskin Bond beautifully said, "Education must inspire the spirit of inquiry, Creativity and Joy" True learning goes beyond memorisation-it encourages curiosity, fosters creativity, and makes the learning process meaningful and enjoyable.

In alignment with the vision of NEP 2020, the CBSE framework now places emphases on competency-based assessments for 50% of the evaluation, highlighting the need for students to develop critical thinking and problem-solving skills. The Support Material is designed to help students analyse concepts deeply, think innovatively, and apply their knowledge affectively, ensuring they are well-prepared not only for exams but also for real-life challenges.

I appreciate the dedicated efforts of the entire team of subject experts in developing this valuable learning resource. I am confident that both teachers and students will make the best use of these material to enhance learning and academic success.

Wishing all students great success in their exam and a bright, fulfilling future ahead.

Veditha
(VEDITHA REDDY, IAS)

Dr. RITA SHARMA
Additional Director of Education
(School/Exam)

सत्यमेव जयते

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D.O. No. DE.5/228/Exam/Message/SM/2018/570
Dated: .. *02/07/2025*

MESSAGE

"Children are not things to be molded, but are people to be unfolded." - Jess Lair

In line with this insightful quote, the Directorate of Education, Delhi, has always made persistent efforts to nurture and unfold the inherent potential within each student. This support material is a testimony to this commitment.

The support material serves as a comprehensive tool to facilitate a deeper understanding of the curriculum. It is crafted to help students not only grasp essential concepts but also apply them effectively in their examinations. We believe that the thoughtful and intelligent utilization of these resources will significantly enhance the learning experience and academic performance of our students.

Our expert faculty members have dedicated themselves to the support material to reflect the latest CBSE guidelines and changes. This continuous effort aims to empower students with innovative approaches, fostering their problem-solving skills and critical thinking abilities.

I extend my heartfelt congratulations to the entire team for their invaluable contribution to creating a highly beneficial and practical support material. Their commitment to excellence ensures that our students are well-prepared to meet the challenges of the CBSE examinations and beyond.

Wishing you all success and fulfilment in your educational journey.

(Dr. Rita Sharma)

DIRECTORATE OF EDUCATION

Govt. of NCT, Delhi

SUPPORT MATERIAL

(2025-2026)

MATHEMATICS

Class : XII

(English Medium)

NOT FOR SALE

PUBLISHED BY : DELHI BUREAU OF TEXTBOOKS

भारत का संविधान

भाग 4क

नागरिकों के मूल कर्तव्य

अनुच्छेद 51 क

मूल कर्तव्य - भारत के प्रत्येक नागरिक का यह कर्तव्य होगा कि वह -

(क) संविधान का पालन करे और उसके आदर्शों, संस्थाओं, राष्ट्रध्वज और राष्ट्रगान का आदर करे;

(ख) स्वतंत्रता के लिए हमारे राष्ट्रीय आंदोलन को प्रेरित करने वाले उच्च आदर्शों को हृदय में संजोए रखे और उनका पालन करे;

(ग) भारत की संप्रभुता, एकता और अखंडता की रक्षा करे और उसे अक्षुण्ण बनाए रखे;

(घ) देश की रक्षा करे और आह्वान किए जाने पर राष्ट्र की सेवा करे;

(ङ) भारत के सभी लोगों में समरसता और समान भ्रातृत्व की भावना का निर्माण करे जो धर्म, भाषा और प्रदेश या वर्ग पर आधारित सभी भेदभावों से परे हो, ऐसी प्रथाओं का त्याग करे जो महिलाओं के सम्मान के विरुद्ध हों;

(च) हमारी सामासिक संस्कृति की गौरवशाली परंपरा का महत्त्व समझे और उसका परिरक्षण करे;

(छ) प्राकृतिक पर्यावरण की, जिसके अंतर्गत वन, झील, नदी और वन्य जीव हैं, रक्षा करे और उसका संवर्धन करे तथा प्राणिमात्र के प्रति दयाभाव रखे;

(ज) वैज्ञानिक दृष्टिकोण, मानववाद और ज्ञानार्जन तथा सुधार की भावना का विकास करे;

(झ) सार्वजनिक संपत्ति को सुरक्षित रखे और हिंसा से दूर रहे;

(ञ) व्यक्तिगत और सामूहिक गतिविधियों के सभी क्षेत्रों में उत्कर्ष की ओर बढ़ने का सतत् प्रयास करे, जिससे राष्ट्र निरंतर बढ़ते हुए प्रयत्न और उपलब्धि की नई ऊँचाइयों को छू सके; और

(ट) यदि माता-पिता या संरक्षक है, छह वर्ष से चौदह वर्ष तक की आयु वाले अपने, यथास्थिति, बालक या प्रतिपाल्य को शिक्षा के अवसर प्रदान करे।

भारत का संविधान

उद्देशिका

हम, भारत के लोग, भारत को एक [1][**संपूर्ण प्रभुत्व-संपन्न समाजवादी पंथनिरपेक्ष लोकतंत्रात्मक गणराज्य**] बनाने के लिए, तथा उसके समस्त नागरिकों को :

सामाजिक, आर्थिक और राजनैतिक **न्याय,**

विचार, अभिव्यक्ति, विश्वास, धर्म

और उपासना की **स्वतंत्रता,**

प्रतिष्ठा और अवसर की **समता**

प्राप्त कराने के लिए,

तथा उन **सब में**

व्यक्ति की गरिमा और [2][**राष्ट्र की एकता**

और अखंडता] सुनिश्चित करने वाली **बंधुता**

बढ़ाने के लिए

दृढ़संकल्प होकर **अपनी इस संविधान सभा में** आज तारीख 26 नवंबर, 1949 ई. को **एतद्द्वारा इस संविधान को अंगीकृत, अधिनियमित और आत्मार्पित करते हैं।**

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1. संविधान (बयालीसवां संशोधन) अधिनियम, 1976 की धारा 2 द्वारा (3.1.1977 से) ''प्रभुत्व-संपन्न लोकतंत्रात्मक गणराज्य'' के स्थान पर प्रतिस्थापित।
2. संविधान (बयालीसवां संशोधन) अधिनियम, 1976 की धारा 2 द्वारा (3.1.1977 से) ''राष्ट्र की एकता'' के स्थान पर प्रतिस्थापित।

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

Review Team
Mathematics (Class XII)
Session-(2025-2026)

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Smt. Suman Arora	Lecturer Mathematics	RPVV, Paschim Vihar
Mr. Pramod Kumar Gupta	Lecturer Mathematics	RPVV, Kishanganj

ANNUAL SYLLABUS
CLASS XII
SUBJECT: MATHEMATICS (041)
SESSION (2025-26)

CONTENT

Unit-I: Relations and Functions

1.Relations and Functions

Types of relations: reflexive, symmetric, transitive and equivalence relations. One to one and onto functions.

2. Inverse Trigonometric Functions

Definition, range, domain, principal value branch. Graphs of inverse trigonometric functions.

Unit-II: Algebra

1.Matrices

Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operation on matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. On- commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. Determinants

Determinant of a square matrix (up to 3 x 3 matrices), minors, co-factors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

Unit-III: Calculus

1.Continuity and Differentiability

Continuity and differentiability, chain rule, derivative of inverse trigonometric functions, *like* $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$, derivative of implicit functions. Concept of exponential and logarithmic functions. Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives.

2.Applications of Derivatives

Applications of derivatives: rate of change of quantities , increasing/decreasing functions, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real- life situations).

3.Integrals

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, Evaluation of simple integrals of the following types and problems based on them

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{px + q}{ax^2 + bx + c} dx$$

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx, \int \sqrt{a^2 \pm x^2} dx, \int \sqrt{x^2 - a^2} dx, \int \sqrt{ax^2 + bx + c}\, dx$$

Fundamental Theorem of Calculus (without proof).Basic properties of definite integrals and evaluation of definite integrals.

4.Applications of the Integrals:

Applications in finding the area under simple curves, especially lines, circles/parabolas/ellipses (in standard form only)

5.Differential Equations

Definition, order and degree, general and particular solutions of a differential equation. Solution of differential equations by method of separation of variables, solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type

Solutions of linear differential equation of the type:

$\frac{dy}{dx} + py = q$ **, where p and q are functions of x or constants**

$\frac{dx}{dy} + px = q$ **,where p and q are functions of y or constants.**

Completion of Mid Term Syllabus by 06th September 2025
Revision for Mid Term Exam
Mid Term Exam
Discussion of Mid Term Question Paper
Unit-IV: Vectors and Three-Dimensional Geometry

1.Vectors

Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and application of scalar (dot) product of vectors, vector (cross) product of vectors.

2.Three - dimensional Geometry

Direction cosines and direction ratios of a line joining two points. Cartesian equation and vector equation of a line, skew lines, shortest distance between two lines. Angle between two lines.

Unit-V: Linear Programming

1.Linear Programming

Introduction, related terminology such as constraints, objective function, optimization, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded or unbounded), feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

Unit-VI: Probability

1.Probability

Conditional probability, multiplication theorem on probability, independent events, total probability, Bayes' theorem .

***Note- Syllabus must be completed by 6th December 2025**

Preparation for Pre Board Examination (2025-26)

Pre Board Examination (2025-26)

CBSE BOARD EXAMINATION

(2025-26)

For relevant NCERT textual material and further information kindly refer to CBSE guidelines
https://cbseacademic.nic.in/

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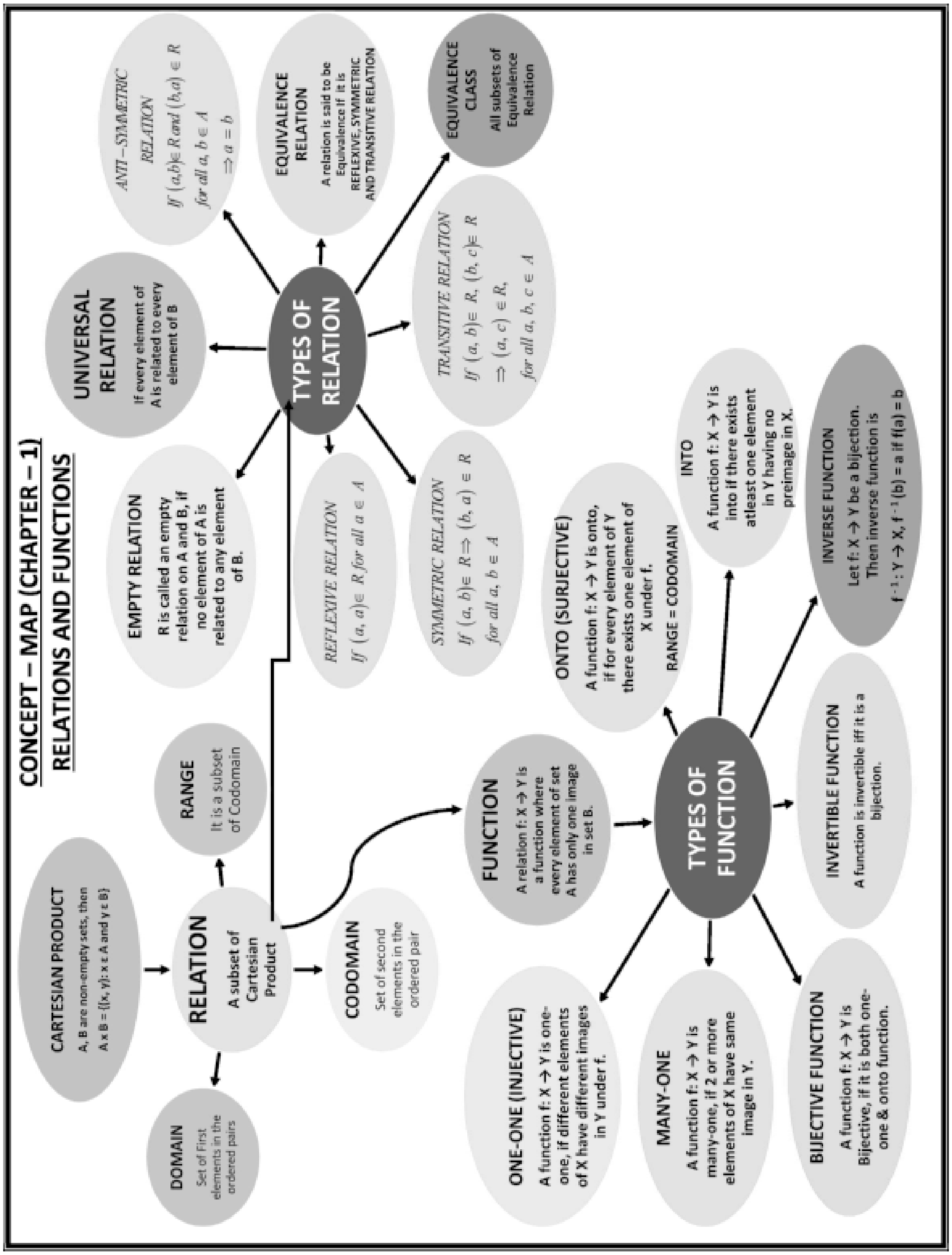
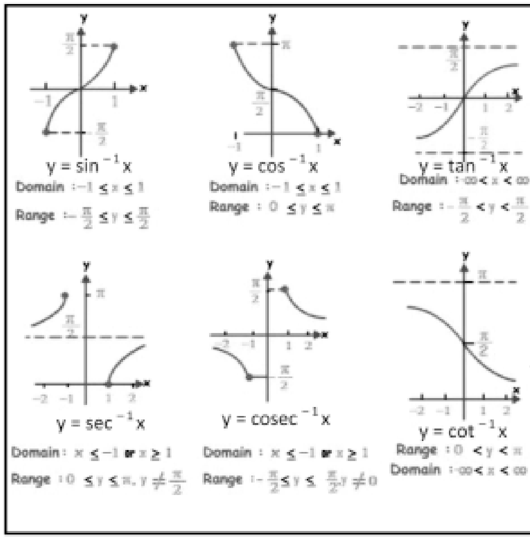

CONCEPT – MAP (CHAPTER – 1)
RELATIONS AND FUNCTIONS
CARTESIAN PRODUCT
A, B are non-empty sets, then A x B = {(x, y): x ε A and y ε B}
RELATION
A subset of Cartesian Product
DOMAIN
Set of First elements in the ordered pairs
RANGE
It is a subset of Codomain
CODOMAIN
Set of second elements in the ordered pair
TYPES OF RELATION
EMPTY RELATION
R is called an empty relation on A and B, if no element of A is related to any element of B.
UNIVERSAL RELATION
If every element of A is related to every element of B
ANTI – SYMMETRIC RELATION
If $(a,b) \in R$ and $(b,a) \in R$ for all $a, b \in A$ $\Rightarrow a = b$
EQUIVALENCE RELATION
A relation is said to be Equivalence if it is REFLEXIVE, SYMMETRIC AND TRANSITIVE RELATION
EQUIVALENCE CLASS
All subsets of Equivalence Relation
TRANSITIVE RELATION
If $(a, b) \in R$, $(b, c) \in R$ $\Rightarrow (a, c) \in R$, for all $a, b, c \in A$
SYMMETRIC RELATION
If $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$
REFLEXIVE RELATION
If $(a, a) \in R$ for all $a \in A$
FUNCTION
A relation f: X → Y is a function where every element of set A has only one image in set B.
TYPES OF FUNCTION
ONE-ONE (INJECTIVE)
A function f: X → Y is one-one, if different elements of X have different images in Y under f.
MANY-ONE
A function f: X → Y is many-one, if 2 or more elements of X have same image in Y.
BIJECTIVE FUNCTION
A function f: X → Y is Bijective, if it is both one-one & onto function.
INVERTIBLE FUNCTION
A function is invertible iff it is a bijection.
INVERSE FUNCTION
Let f: X → Y be a bijection. Then inverse function is $f^{-1}: Y \to X$, $f^{-1}(b) = a$ if $f(a) = b$
INTO
A function f: X → Y is into if there exists atleast one element in Y having no preimage in X.
ONTO (SURJECTIVE)
A function f: X → Y is onto, if for every element of Y there exists one element of X under f.
RANGE = CODOMAIN

CONCEPT – MAP (CHAPTER – 2)
INVERSE TRIGONOMETRIC FUNCTIONS

INVERSE TRIGONOMETRIC FUNCTIONS



- $\sin^{-1}x = \operatorname{cosec}^{-1}(\frac{1}{x}),\ when\ x \in [-1,1]$
- $\cos^{-1}x = \sec^{-1}(\frac{1}{x}),\ when\ x \in [-1,1]$
- $\tan^{-1}x = \cot^{-1}(\frac{1}{x}),\ when\ x \in R$
- $\cot^{-1}x = \tan^{-1}(\frac{1}{x}),\ when\ x \in R$
- $\sec^{-1}x = \cos^{-1}(\frac{1}{x}),\ when\ x \in R-(-1,1)$
- $\operatorname{cosec}^{-1}x = \sin^{-1}(\frac{1}{x}),\ when\ x \in R-(-1,1)$

- $\sin^{-1}(\sin x) = x,\ when\ x \in [-\frac{\pi}{2},\frac{\pi}{2}]$
- $\cos^{-1}(\cos x) = x,\ when\ x \in [0,\pi]$
- $\tan^{-1}(\tan x) = x,\ when\ x \in (-\frac{\pi}{2},\frac{\pi}{2})$
- $\cot^{-1}(\cot x) = x,\ when\ x \in (0,\pi)$
- $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x,\ when\ x \in [-\frac{\pi}{2},\frac{\pi}{2}]-\{0\}$
- $\sec^{-1}(\sec x) = x,\ when\ x \in [0,\pi]-\{\frac{\pi}{2}\}$

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2},\ when\ x \in [-1,1]$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2},\ when\ x \in R$
- $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2},\ when\ x \in R-(-1,1)$

- $\sin(\sin^{-1}x) = x,\ when\ x \in [-1,1]$
- $\cos(\cos^{-1}x) = x,\ when\ x \in [-1,1]$
- $\tan(\tan^{-1}x) = x,\ when\ x \in R$
- $\cot(\cot^{-1}x) = x,\ when\ x \in R$
- $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x,\ when\ x \in R-(-1,1)$
- $\sec(\sec^{-1}x) = x,\ when\ x \in R-(-1,1)$

Inverse trigonometric functions are the inverse functions of the basic trigonometric functions which are sine, cosine, tangent, cotangent, secant and cosecant functions.

- $\sin^{-1}(-x) = -\sin^{-1}(x),\ when\ x \in [-1,1]$
- $\cos^{-1}(-x) = \pi - \cos^{-1}(x),\ when\ x \in [-1,1]$
- $\tan^{-1}(-x) = -\tan^{-1}(x),\ when\ x \in R$
- $\cot^{-1}(-x) = \pi - \cot^{-1}(x),\ when\ x \in R$
- $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x),\ when\ x \in R-(-1,1)$
- $\sec^{-1}(-x) = \pi - \sec^{-1}(x),\ when\ x \in R-(-1,1)$

CONCEPT – MAP (CHAPTER – 3)

MATRICES

Matrices are defined as a rectangular arrangement of numbers or functions,so it is 2-dimensional.

A two-dimensional matrix consists of the number of rows (m) and a number of columns (n). Horizontal ones are called Rows and Vertical ones are called Columns.

ORDER OF MATRIX

The order of matrix is a relationship with the number of elements present in a matrix.

The order of a matrix is denoted by $m \times n$, where m and n are the number of Rows and Columns Respectively and the number of elements in a matrix will be equal to the product of m and n.

Properties of transpose:

$(a)\ (A^T)^T = A$

$(b)\ (kA)^T = kA^T\ (k = \text{constant})$

$(c)\ (A+B)^T = A^T + B^T$

$(d)\ (AB)^T = B^T.A^T$

MATRICES

If $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}_{3\times2}$ then transpose of A donated A^T or A'

and $A^T = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}_{2\times3}$

[order of $A = m \times n$, then order of $A^T = n \times m$]

Symmetric Matrix

A square matrix $A = [a_{ij}]$ is symmetric if $\boxed{A^T = A}$ i.e. $a_{ij} = a_{ji} \forall i$ and j

For Example:

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

- ROW MATRIX, $A = (1\ \ 2\ \ 3)_{1\times3}$
- COLUMN MATRIX, $A = \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{3\times1}$
- SQUARE MATRIX, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times2}$
- ZERO / NULL MATRIX

$$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- DIAGONAL MATRIX, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(*Diagonal* elements may be zero)

- SCALAR MATRIX, $S = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
- IDENTITY MATRIX, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Skew – Symmetric Matrix

A square matrix $A = [a_{ij}]$ is Skew-Symmetric if $\boxed{A^T = -A}$ i.e. $a_{ij} = -a_{ji} \forall i$ and j

(All diagonal elements are zero)

For Example:

$$A = \begin{pmatrix} 0 & -b & -c \\ b & 0 & e \\ c & -e & 0 \end{pmatrix}$$

CONCEPT – MAP (CHAPTER – 4)

DETERMINANTS

DETERMINANTS

Every square matrix associates to an expression or a number which is known as Determinant.

If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by $\det(A)$ *or* $|A|$ *or* Δ.

A determinant of order 2 is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

where a, b, c, d are complex numbers. The value of a determinant may be positive, negative or zero.

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *is given by*

$$A = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| \text{ (sq. units)}$$

MINOR : If we take an element of the determinant and delete / remove the row and column containing that element, the determinant of the elements left is called the minor of that element.

It is denoted by Mij.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} e & f \\ q & r \end{vmatrix}$$

(*Minor of* $a_{11} = M_{11}$)

PROPERTY : If we multiply the elements of any row / column with their respective Cofactors of the other row / Column, then we get zero as a result.

For example,

$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$

$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

COFACTOR : Cofactor of the element aij is $C_{ij} = (-1)^{i+j} M_{ij}$ *; Where i & j denotes the row & column in which the particular element lies*

Magnitude of Minor and Cofactor of a_{ij} *are equal*

$C_{12} = -M_{12}, C_{21} = -M_{21}, C_{23} = -M_{23}$

$C_{11} = M_{11}, C_{22} = M_{22}, C_{13} = M_{13}$

ADJOINT OF A MATRIX

Let $A = [a_{ij}]_{m \times n}$ *be a square matrix and* C_{ij} *be cofactor of* a_{ij} *in* $|A|$. *Then,*

$(adj\ A) = [C_{ij}]$

- $A.(adj\ A) = (adj\ A).A = |A|I$
- $(adj\ AB) = (adj\ B).(adj\ A)$
- $|adj\ A| = |A|^{n-1}$,

where n is the order of a Matrix A

PROPERTY : If we multiply the elements of any row / column with their respective Cofactors of the same row / Column, then we get the value of the determinant.

For example,

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$|A| = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

Note that the value of a determinant of order three in terms of 'Minor' & 'Cofactor' can be written as : $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

OR

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

CONCEPT – MAP (CHAPTER – 4)
DETERMINANTS

DETERMINANTS

A square matrix A is said to be invertible if there exists a square matrix B of the same order such that AB = BA = I then we write $A^{-1} = B$.

$\left(A^{-1} \text{ exists only if } |A| \neq 0\right)$

$$A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

INVERSE OF 2×2 MATRIX

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus, If $A = \begin{pmatrix} 2022 & 1 \\ 2021 & 1 \end{pmatrix}$

then, $A^{-1} = \begin{pmatrix} 1 & -1 \\ -2021 & 2022 \end{pmatrix}$

SINGULAR MATRIX

A Matrix A is singular if $|A| = 0$ and it is non–singular if $|A| \neq 0$,

So, $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5 \neq 0$

A is Non–Singular Matrix.

$|B| = \begin{vmatrix} 2 & 8 \\ 1 & 4 \end{vmatrix} = 8 - 8 = 0,$

So B is Singular Matrix.

PROPERTIES OF A^{-1}:

- $(AB)^{-1} = B^{-1}.A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $A.A^{-1} = A^{-1}A = I$
- $|A^{-1}| = \frac{1}{|A|}$
- $|A.adjA| = |A|^n$

(Where n is the order of Matrix A)

Solution of system of Linear Equations Using Matrix method

Consider $a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Then, we can write these equations as

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \Rightarrow AX = B$$

Thus, Unique solution is given by $X = A^{-1}B$, when $|A| \neq 0$.

A system of equation $AX = B$ is said to be consistent or inconsistent according as its solution exists or not.

For a square matrix A in matrix equation $AX = B$

- If $|A| \neq 0$, there exists a unique solution and system of equations is consistent.
- If $|A| = 0$ & $(adjA).B \neq 0$, there exists no solution and system of equations is inconsistent.
- If $|A| = 0$ & $(adjA).B = 0$, then system may or may not be consistent according as the system has infinitely many solution or no solution.

5

CONCEPT MAP OF CONTINUITY AND DIFFERENTIABILITY

CONTINUITY

Continuity of a function at a point -

Suppose f is a rael function on a subset of real numbers & let c be a point in the domain of f . then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

Continuity of a function in an interval

Suppose f is a function defined on a closed interval [a,b], then for f to be continuous it needs to be continuous at every point in [a,b]includint the end points a & b

Continuity of f at a, $\lim_{x \to a^+} f(x) = f(a)$

Continuity of f at b, $\lim_{x \to b^-} f(x) = f(b)$

A function which is not continuous at x=c is said to be discontinuous at that point

Differentiation of Inverse Trigonometric Functions

$f(x)$	$f'(x)$	Domain of f
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	(-1,1)
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	(-1,1)
$\tan^{-1} x$	$\frac{1}{1+x^2}$	R
$\cot^{-1} x$	$\frac{-1}{1+x^2}$	R
$sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$	$\lvert x \rvert > 1$
$cosec^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}$	$\lvert x \rvert > 1$

Algebra of Continuous Function

Theorem 1:Suppose f & g be two functions continuous at a real numbe c , Then
(1)f+g is continuous at x=c
(2)f-g is continuous at x=c
(3)f.g is continuous at x=c
(4)f/g is continuous at x=c (provided g(c)≠0)
Theorem 2; Suppose f & g are real valued functions such that
Fog is defined at c .If g is continuous at c
& if f is Continuous at g(c) the (fog) is continuous at c

Implicit Functions

An equation in the form f(x,y)=0 in which y is not expressible in terms of x is called implicit function of x &y

Derivative of Implicit Functions

Let y=f(x,y) where f(x,y) is an implicit function of x&y
*Firstly differentiate both sides of equation w.r.t x.
*Then take all terms involving dy/dx on LHS & remaining terms on R H S to get the required value .

Differentiability

A function is said to be differentiable at a point c in its domain if its left hand And right hand derivatives exists at C and are equal

Here at x=c , left Hand Derivative L.H. D= $\lim_{h \to 0} \frac{f(c-h)-f(c)}{-h} = Lf'(c)$

Right Hand Derivative R.H.D= $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = Rf'(c)$

Theorem: If a fuction f is differntiable at a point c then it is also continuous at that point therefore every differentiable function is continuous but converse is not true

Chain Rule

If y is a function of u
u is a function of v, &
v is a function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

Derivative of a Function in Parametric Form

The set of equations x=f(t), y=g(t) is Called parametric form of an equation
Here dy/dx=(dy/dt)/ dx/dt=g(t)/f(t)
Here dy/dx is expressed in terms of parameters only Without directly involving the main variable.

Algebra Of Derivatives

Let u v be the function of x

(1)Sum and Differnce rule

$$(u \pm v)' = (u' \pm v')$$

(2) Leibnitz or product rule (uv)'=u'v+vu'

(3) Quotient Rule $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$

Second Order Derivative

Let y=f(x) then
If f(x) is differentiable
then we may differentiate it
again w.r.t x & get the second order
derivative represented by
or f''(x) or D''yor y''or y_2 $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

CONCEPT MAP OF CONTINUITY AND DIFFERENTIABILITY

Noteworthy Results on Continuous Functions
* A constant Function f(x)=k is continuous everywhere.
* Identity Function f(x)=x is continuous everywhere.
* Polynomial Function f(x)= $f(x)=a_0+a_1x+a_2x^2+..........a_nx^n, n\in N, x\in R$ is continuous everywhere.
* The modulus function f(x)=|x| is continuous everywhere.
* The logarithmic function f(x)=x is continuous in its domain
* The exponential function f(x)= $a^x, a>0$ is continuous everywhere.
* The sine function f(x)=sinx and cosine function f(x) =cosx are everywhere continuous .
*The tangent function, cotangent function, secant function and cosecant function are continuous in their respective domains.
*All the six inverse trigonometric functions are continuous in their respective domains.
*A rational function f(x)=g(x)/h(x) , h(x)not equal to zero is continuous at every point of its domain.
* Sum , difference ,product and quotient of of two continuous function is a continuous function.

A function f may fail to be continuous at x=0 for any of the following reasons

(1) f is not defined at x=a, i.e,f(a) does not exist

(2)Either $\lim_{x\to a^-} f(x)$ does not exist or $\lim_{x\to a^+} f(x)$ does not exist.

(3) $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

(4) $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$

6 APPLICATION OF DERIVATIVE

Rate of Change of Quantities

If a quantity if 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx}\left[f'(x)\right]$ represents the rate of change of y w.r.t x and $\left.\frac{dy}{dx}\right|_{x=x_0}$ $(f'(x_0))$ represents the rate of change of y w.r.t. x at $x = x_0$

If 'x' and 'y' varies with another variable 't' i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

For eg: if the radius of a circle, r = 5 cm, then the rate of change of the area of a circle per second w.r.t 'r' is - $\left.\frac{da}{dr}\right|_{r=5} = \frac{d}{dr}(\pi r^2)\Big|_{r=5} = 2\pi r\big|_{r=5} = 10\pi$

Maxima & Minima

A point C in the domain of 'f' at which either $f'(c)=0$ or is not differentiable is called a critical point of f.

First Derivative Test

Let f be continuous at a critical point C in open I. Then (i) if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. (ii) If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.
(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second Derivative Test

Let f be a function defined on I and C∈I, f is twice differentiable at C. Then
(i) x=C is a point of local max. If $f'(C)=0$ and $f''(C) < 0$, $f(C)$ is local max. of f.
(ii) x=C is a point of local min if $f'(C)=0$ and $f''(C)>0$. $f(C)$ is local min of f. (iii) The test fails if $f'(C)=0$ and $f''(C)=0$

Increasing & Decreasing Functions

A function f is said to be (i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \le f(x_2) \forall x_1, x_2 \in (a,b)$, and (ii) decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) > f(x_2) \forall x_1, x_2 \in (a,b)$

If $f'(x) \ge 0 \forall x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \le 0 \forall x \in (a,b)$, then f is decreasing in (a,b) For eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall x \in R$.
So, the function f is strictly increasing on R.

7 CONCEPT MAP OF INTEGRAL

INTEGRATION BY SUBSTITUTION

The method in which we change the variable to some other variable is called the method of substitution

$\int \tan x\,dx = \log|\sec x| + c$ $\quad \int \cot x\,dx = \log|\sin x| + c$

$\int \sec x\,dx = \log|\sec x + \tan x| + c$ $\quad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c.$

INDEFINITE INTEGRAL

It is the inverse of differentiation. Let $\frac{d}{dx}F(x) = f(x)$. Then $\int f(x)dx = F(x) + c$, 'c': constant of integral. These integrals are called indefinite or general integrals.

Properties of indefinite integrals are

(i) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$, $\quad$ (ii) $\int kf(x)dx = k\int f(x)dx$,

For eg : $\int (3x^2 + 2x)dx = x^3 + x^2 + c$ where k is real.

INTEGRATION OF SOME SPECIAL FUNCTIONS

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ $\quad$ (ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(iii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ $\quad$ (iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

(v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$ $\quad$ (vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c.$

SOME STANDARD INTEGRALS

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$ like, $\int dx = x + c$

(ii) $\int \cos x\,dx = \sin x + c$ $\quad$ (iii) $\int \sin x\,dx = -\cos x + c$

(iv) $\int \sec^2 x\,dx = \tan x + c$ $\quad$ (v) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

(vi) $\int \sec x \tan x\,dx = \sec x + c$ $\quad$ (vii) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ $\quad$ (ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$ $\quad$ (xi) $\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$

(xii) $\int e^x dx = e^x + c$ $\quad$ (xiii) $\int a^x dx = \frac{a^x}{\log a} + c$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$ $\quad$ (xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + c$

(xvi) $\int \frac{1}{x}dx = \log|x| + c$

INTEGRATION BY PARTS

$$\int f_1(x) f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int \frac{d}{dx}f_1(x)\int f_2(x)\,dx$$

INTEGRATION BY PARTIAL FRACTIONS

A rational function of the form $\frac{P(x)}{Q(x)}(Q(x) \neq 0) = T(x) + \frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b.$

(ii) $\frac{px+q}{(x+a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

SOME SPECIAL TYPE OF INTEGRALS

(i) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c.$

(ii) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c.$

(iii) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$

FIRST FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let the area function be defined by $A(x) = \int_a^x f(x)dx \; \forall x \geq a$, where f is continuous on $[a, b]$ then $A'(x) = f(x) \; \forall x \in [a, b]$.

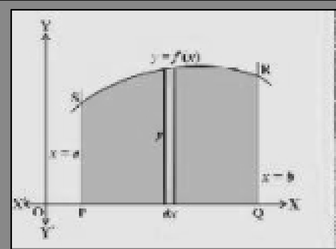
SECOND FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let f be a continuous function of x defined on $[a,b]$ and let F be another function such that $\frac{d}{dx}F(x) = f(x) \forall x \in$ domain of f, then $\int_a^b f(x)dx = [F(x) + c]_a^b = F(b) - F(a)$. This is called the definite integral of f over the range $[a,b]$, where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

CONCEPT – MAP (CHAPTER – 8)
APPLICATIONS OF INTEGRALS

APPLICATIONS OF INTEGRALS

Area of the regions bounded by simple curves

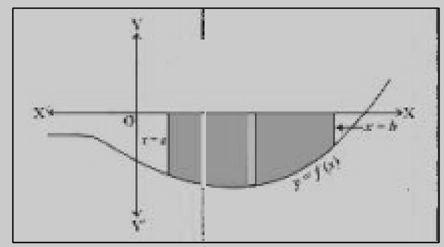


(A) *The area bounded by the curve* $y = f(x)$ *lies above the* $X-$*axis and the ordinates* $x = a$ *and* $x = b$ *is given by*

$$Area = \int_a^b y\,dx = \int_a^b f(x)dx$$

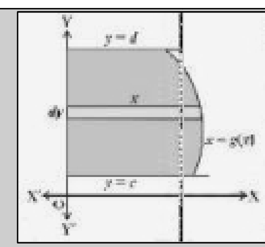
(B) *The area bounded by the curve* $y = f(x)$ *lies below the* $X-$*axis and the ordinates* $x = a$ *and* $x = b$ *is given by*

$$Area = -\int_a^b y\,dx = \left|\int_a^b y\,dx\right| = \left|\int_a^b f(x)dx\right|$$



(C) *The area bounded by the curve* $x = f(y)$*, lies right* $Y-$*axis and abscissae* $y = c$ *and* $y = d$ *is given by*

$$Area = \int_c^d x\,dy = \int_c^d g(y)dy$$

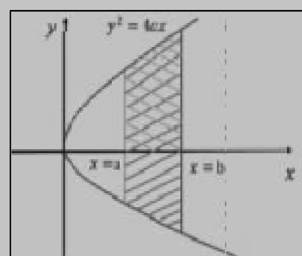


CONCEPT – MAP (CHAPTER – 8)
APPLICATIONS OF INTEGRALS

APPLICATIONS OF INTEGRALS

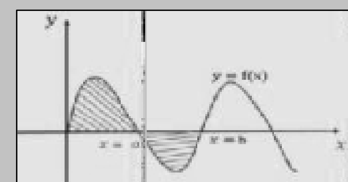
Symmetrical Area

If the curve is symmetrical about a coordinate axis (x axis , y axis , origin, a line), then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.



Positive and Negative area

Positive and Negative area: Area is always taken as positive. If some part of the area lies in the + ve side i.e., above X – axis and some part lies in the – ve i.e., below X – axis, then the area of two parts should be calculated separately and then add their numerical values to get the desired area.



$$Area = \int_0^a y dx + \left| \int_a^b y\, dx \right|$$

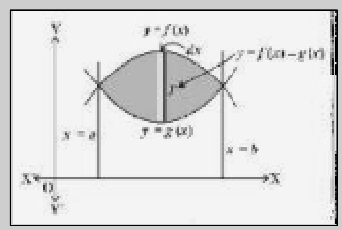
Area between two curves

Area between two curves:

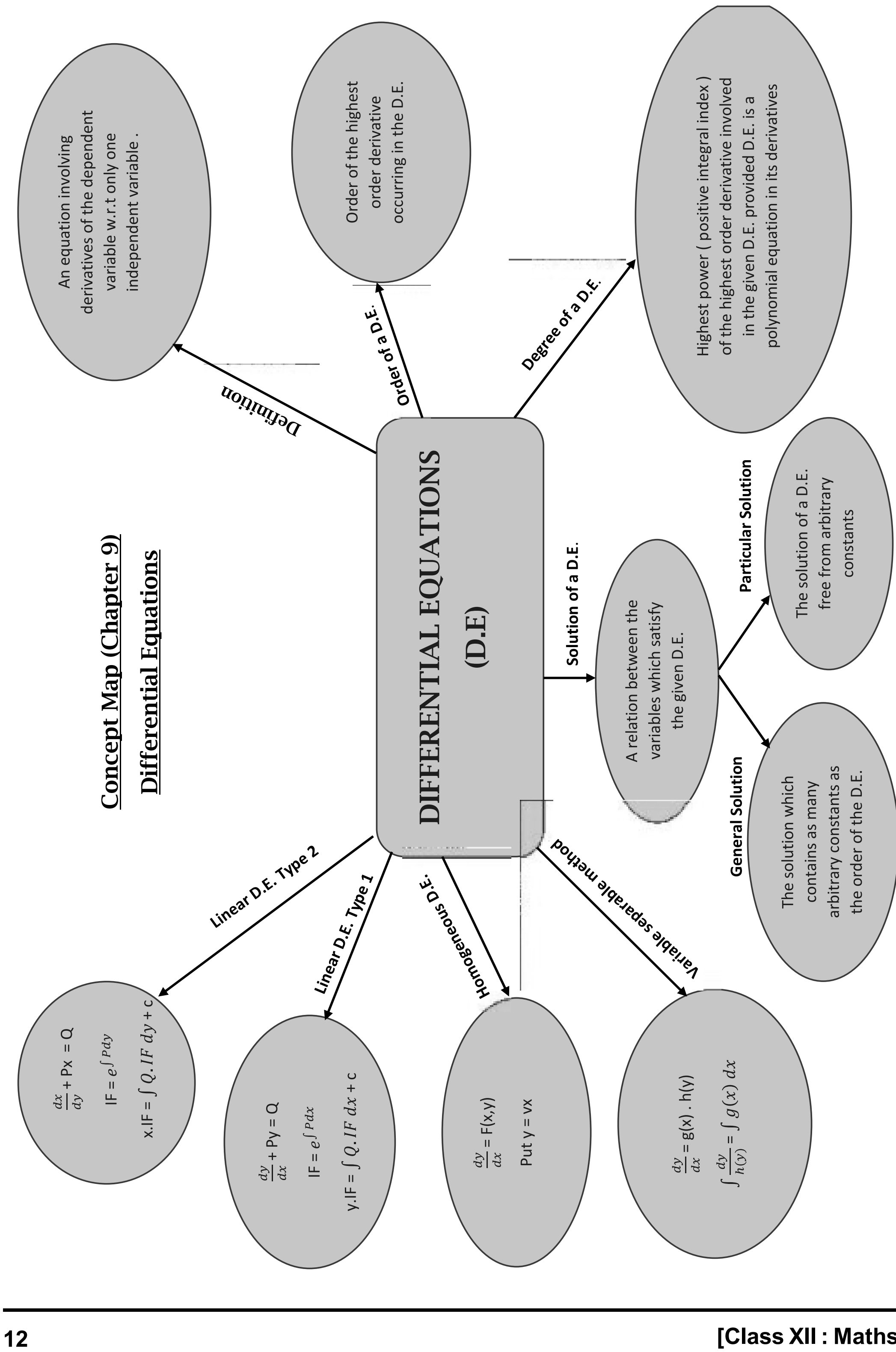
When both curves intersect at two points and their common area lies between these points

If the curves $y_1 = f(x)$ & $y_2 = g(x)$ *intersect at two points* $A(x = a)$ *and* $B(x = b)$, then the area between the curves is given by

$$Area = \int_a^b (y_1 - y_2)dx = \int_a^b (f(x) - g(x))$$



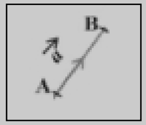
Concept Map (Chapter 9)
Differential Equations



CONCEPT – MAP (CHAPTER – 10)

VECTORS

A quantity that has magnitude as well as direction is called a vector.



A directed line segment is a vector denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

If a point P in space, having coordinates (x, y, z) with respect to the origin O $(0, 0, 0)$. Then, the vector $\overrightarrow{OP}$ having O and P as its initial & terminal points, respectively, is called the position vector of the point P with respect to O.

Using distance formula, the magnitude of $\overrightarrow{OP}$ $(or\ \vec{r})$ is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

The angles made by $\overrightarrow{OP}$ with positive direction of x, y & z-axes (say α, β & γ respectively) are called *its direction* angles, and the cosine value of these angles i.e. $\cos\alpha, \cos\beta$ & $\cos\gamma$ are called direction cosines of $\overrightarrow{OP}$ *denoted by l, m & n respectively.*

VECTORS

TYPES OF VECTORS

- *A zero vector is a vector when the magnitude of the vector is zero and the starting point of the vector coincides with the terminal point.*
- *A vector which has a magnitude of unit length is called a unit vector.*
- *Two or more vectors which have the same starting point are called co–initial vectors.*
- *Two vectors are collinear if they are parallel to the same line irrespective of their magnitudes and direction.*
- *Two or more vectors are said to be equal when their magnitude is equal and also their direction is the same.*
- *Negative of a Vector: If two vectors are the same in magnitude but exactly opposite in direction then both the vectors are negative of each other.*

Vector Joining Two Points

Let $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ be any two points in the space, then $\overrightarrow{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ &

$\overrightarrow{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

SCALAR(DOT) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ, *then* scalar product is defined as

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta, 0 \le \theta \le \pi$

VECTOR(CROSS) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ, *then* vector product is defined as

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{n}$, *where* $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}$ *and* $\vec{b}$.

SECTION FORMULAE

The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\vec{a}$ & $\vec{b}$ respectively, in the ratio $m : n$

(i) Internally is given by $\dfrac{m\vec{b} + n\vec{a}}{m + n}$

(ii) Externally is given by $\dfrac{m\vec{b} - n\vec{a}}{m - n}$

The position vector of middle point of PQ is given by $\dfrac{\vec{a} + \vec{b}}{2}$.

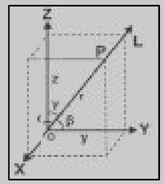
CONCEPT – MAP (CHAPTER – 11)
THREE DIMENSIONAL GEOMETRY

THREE DIMENSIONAL GEOMETRY

DIRECTION COSINES OF A LINE (DC'S)

The direction cosines are denoted by l, m, n.

Thus, $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$



- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
- $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

DIRECTION RATIO'S OF A LINE (DR's)

Any three numbers a, b and c proportional to the direction cosines l, m and n respectively are called direction Ratios of the line.

- *The Direction ratios of a line passing through two points* $A(x_1, y_1, z_1)$ *and* $B(x_2, y_2, z_2)$ *are* $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$.
- $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
- $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

EQUATION OF A LINE

- *Equation of a line through a given point with position vector* $\vec{a}$ *and parallel to a given vector* $\vec{b}$

<u>*VECTOR FORM :*</u> $\vec{r} = \vec{a} + \lambda\vec{b}$

CARTESIAN FORM : $\frac{x - x_1}{p} = \frac{y - y_1}{q} = \frac{z - z_1}{s}$

Where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},$

$\vec{b} = p\hat{i} + q\hat{j} + s\hat{k}$

NOTE : $< p, q, s >$ *are d.r.'s of the line*

SHORTEST DISTANCE BETWEEN TWO SKEW – LINES

Let the lines are $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ & $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, *then*

$S.D. = \left|\frac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|$

DISTANCE BETWEEN TWO PARALLEL – LINES

Let the lines are $\vec{r} = \vec{a_1} + \lambda\vec{b}$ & $\vec{r} = \vec{a_2} + \mu\vec{b}$, *then*

$S.D. = \left|\frac{(\vec{a_2} - \vec{a_1}) \times \vec{b}}{|\vec{b}|}\right|$

EQUATION OF A LINE

- *Equation of a line through passing through* two *given point with position vector* $\vec{a}$ *and* $\vec{b}$

<u>*VECTOR FORM :*</u> $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

CARTESIAN FORM : $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

Where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

ANGLE BETWEEN TWO LINES

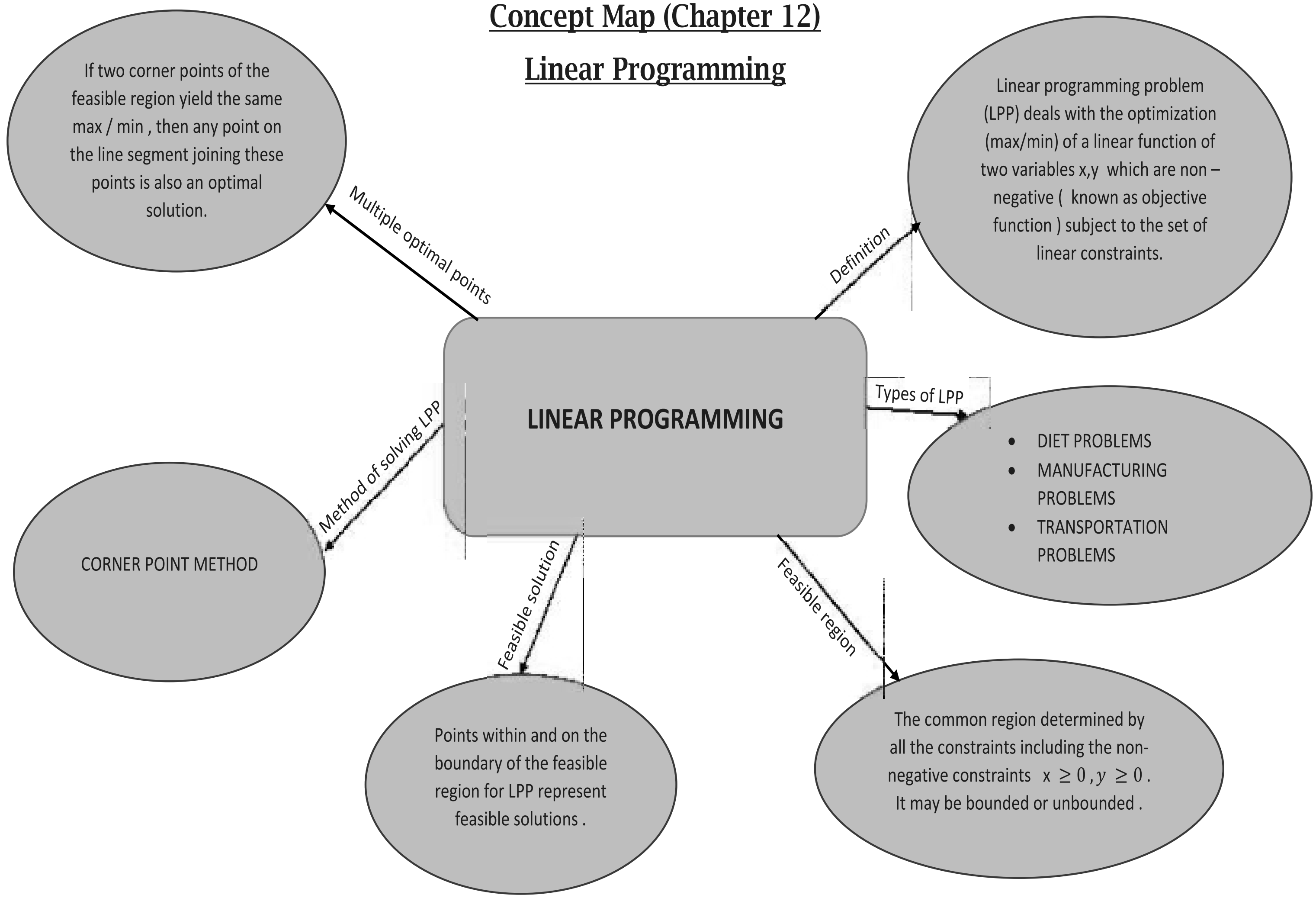
$L_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ & $L_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ *is*

$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right| = |l_1l_2 + m_1m_2 + n_1n_2|$

- *If two lines are perpendicular then* $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- *If two lines are parallel then* $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

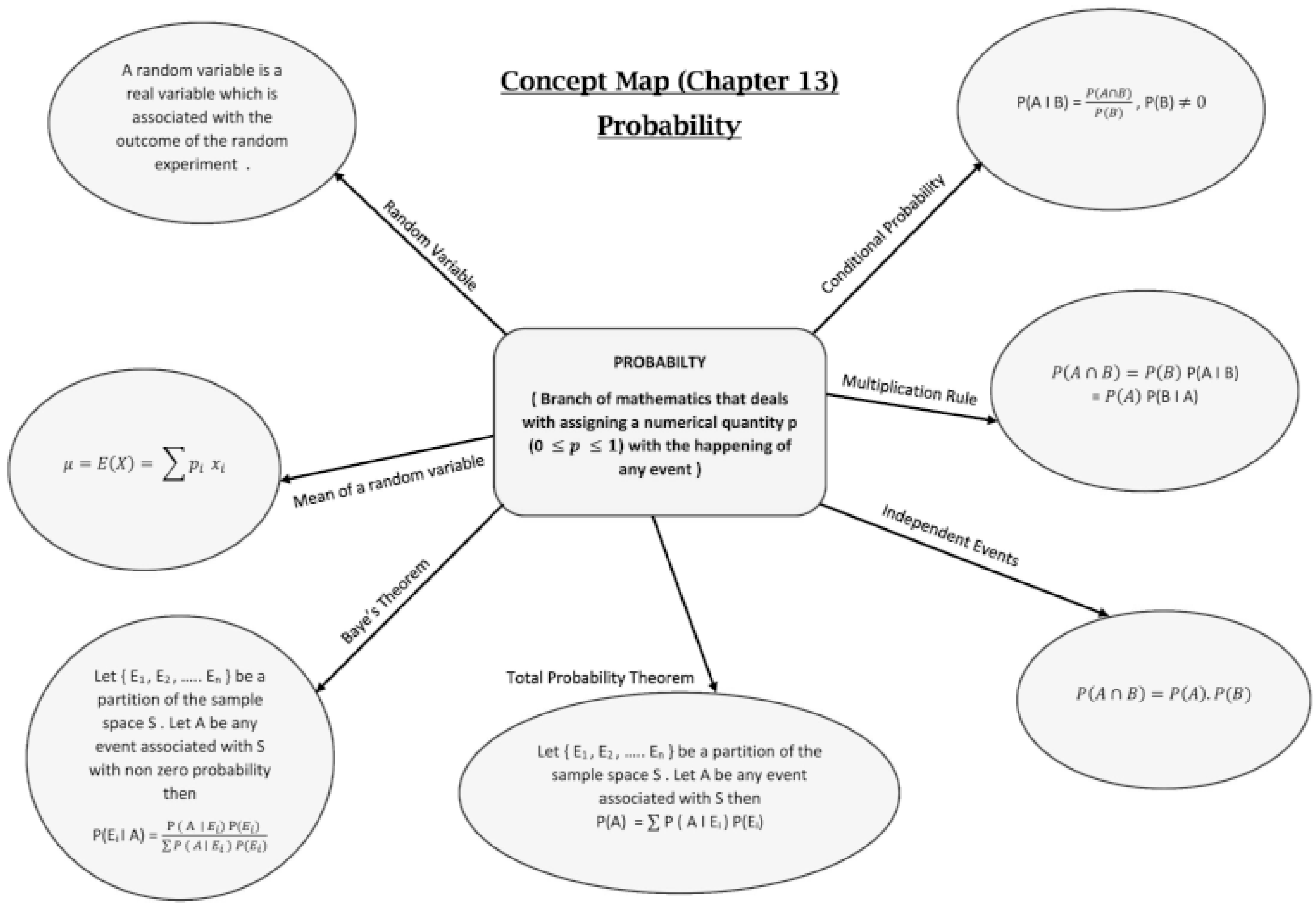
Concept Map (Chapter 12)

Linear Programming



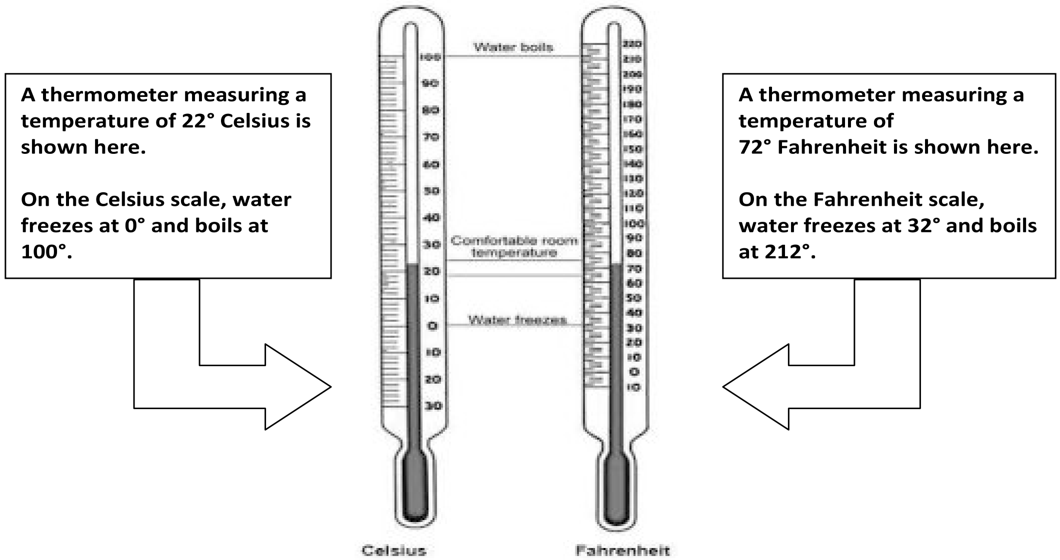
Concept Map (Chapter 13)

Probability



CHAPTER-1

RELATIONS AND FUNCTIONS



By looking at the the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50°F and 80°F (or between 10°C and 25°C). If a meteorologist predicts an average temperature of 0°C (or 32°F), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenhelt measurement or vice versa.

For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process. Using the function

$F = f(C) = \frac{9}{5}C + 32$, any temperature in Celsius can be converted into Fahrenheit scale.

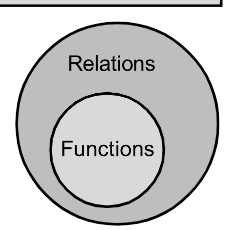
TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM 2025-26
Types of relations: reflexive, symmetric, transitive and equivalence relations.
One to one and onto functions

A relation in a set A is a subset of A × A.

Thus, R is a relation in a set A = $R \subseteq A \times A$

If (a, b) ε R then we say that a is related to b and write, a R b

If (a, b) $\notin$ in R then we say that a is not related to b and write, a $\not R$ b.



If number of elements in set A and set B are p and q respectively, Means $n(A) = p$, $n(B) = q$, then

No. of Relation af A × A = 2^{p^2}

No. of Relation of B × B = 2^{q^2}

No. of Relation of A × B = No. of Relation of B × A = 2^{pq}

No. of **NON EMPTY** Relation of A × A = $(2^{p^2} - 1)$,

No. of **NON EMPTY** Relation of B × B = $(2^{q^2} - 1)$.

No. of **NON-EMPTY** Relation of A × B = No. of Relation of B × A = $(2^{pq} - 1)$

Q.1 If A = {*a, b, c*} and B = {1, 2} find the number of Relation R on (i) A × A (ii) B × B (iii) A × B

Ans. As $n(A) = 3$, $n(B) = 2$, so

No. of Relation R on A × A = $2^{3 \times 3} = 2^9 = 512$

No. of Relation R on B × B = $2^{2 \times 2} = 2^4 = 16$

No. of Relation R on A × B = $2^{3 \times 2} = 2^6 = 64$

Q.2 A = {*d*, *o*, *e*} and B = {22, 23} find the number of Non-empty Relation R on (i) A × A (ii) B × B

Ans. As $n(A) = 3$, $n(B) = 2$, so

No. of Non-empty relations R on A × A = $2^{3 \times 3} - 1 = 2^9 - 1 = 511$

No. of Non-empty relations R on B × B = $2^{2 \times 2} - 1 = 2^4 - 1 = 15$

Different types of relations

- **Empty Relation Or Void Relation**

  A relation R in a set A is called an empty relation, if no element of A is related to any element of A and we denote such a relation by ϕ.

  Example: Let A = {1, 2, 3, 4} and let R be a relation in A, given by R = {(*a, b*): $a + b = 20$}.

- **Universal Relation**

  A relation R in a set A is called an universal relation, if each element of A is related to every element of A.

  Example: Let A = {1, 2, 3, 4} and let R be a relation in A, given by R = {(*a, b*): $a + b > 0$}.

- **Identity Relation**

  A relation R in a set A is called an identity relation, where R = {(*a, a*), $a \in A$}.

  Example : Let A = {1, 2, 3, 4} and let R be a relation in A, given by R = {(1, 1), (2, 2), (3, 3), (4, 4)}.

- **Reflexive Relation**

  A relation R in a set A is called a Reflexive relation, if $(a, a) \in R$, for all $a \in A$.

  Example : Let A = {1, 2, 3, 4} and let R be a relation in A, given by

  R = {(1, 1), (2, 2), (3, 3), (4, 4)}.

  R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)}.

  R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (1, 3), (3, 1)}.

- **Symmetric Relation**

  A relation R in a set A is called a symmetric relation, if $(a, b) \in R$, then $(b, a) \in R$ for all $a, b \in A$.

  Example : Let A = {1, 2, 3, 4} and let R be a relation in A, given by

  R = {(1, 1), (2, 2), (3, 3)}.

  R = {(1, 2), (2, 1), (3, 3)}.

  R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (1, 3), (3, 1), (3, 2)}.

- **Transitive Relation**

  A relation R in a set A is called a transitive relation,

  if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ for all $a, b, c \in A$

  Or

  $(a, b) \in R$ and $(b, c) \notin R$ for all $a, b, c \in A$

  Example : Let {1, 2, 3, 4} and let R be a relation in A, given by

  R = {(1, 1), (2, 2), (3, 3)}. (According to second condition)

  R = {(1, 2), (2, 1), (1, 1), (2, 2)}. (According to first condition)

  R = {(2, 3), (1, 3), (3, 1), (3, 2), (3, 3), (2, 2), (1, 1)}.

- **Equivalence Relation**

  A relation R in a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Illustration:

Let A be the set of all integers and let R be a relation in A, defined by R = {a, b}: $a = b$}, Prove that R is Equivalence Relation.

Solution: Reflexivity : Let R be reflexive $\Rightarrow (a, a) \in R \ \forall \ a \in A$

$\Rightarrow \quad a = a$, which is true

Thus, R is Reflexive Relation.

Symmetricity : Let $(a, b) \in R \ \forall a, b \in A$

$\Rightarrow \quad a = b$

$\Rightarrow \quad b = a$

so $(b, a) \in R$. Thus R is symmetric Relation.

Transitivity : Let $(a, b) \in$ R and $(b, c) \in R\ \forall\ a, b, c \in$ A

$\Rightarrow\quad a = b$ and $b = c$

$\Rightarrow\quad a = b = c$

$\Rightarrow\quad a = c$

so $(a, c) \in$ R. Thus R is transitive Relation.

As, R is reflexive, Symmetric and transitive Relation

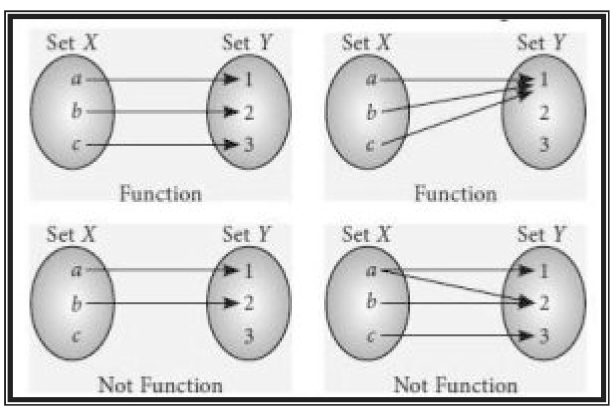
$\therefore$ R is an Equivalence Relation

FUNCTIONS

Functions can be easily defined with the help of concept mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f : X \to Y$.

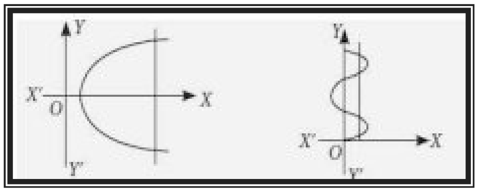
where $y = f(x)$, $x\ \varepsilon$ X and $y\ \varepsilon$ Y. We say that 'y' is the imagesof 'x' under f (or x is the pre image of y).

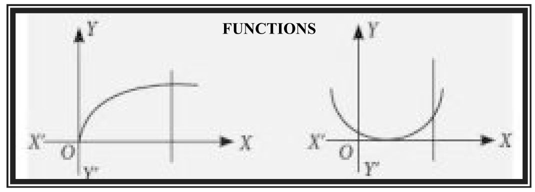
- A mapping f : X $\to$ Y is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.
- Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X.
- Functions cannot be multi-valued (A mapping that is multi-valued is called a relation from X and Y) eg.



Testing for a function by Vertical line Test

A relation f : A $\to$ B is a function or not, it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more thatn one point then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function. Following figures represents which is not a function and which is a function.





NOT FUNCTION NOT A FUNCTION

Number of Functions

Let X and Y be two finite sets having m and n elements respectively. Thus each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

Real valued function: if R, be the set of real numbers and A, B are subsets of R, then the function f : A $\rightarrow$ B is called a real function or real valued functions.

Domain, Co-Domain And Range of Function

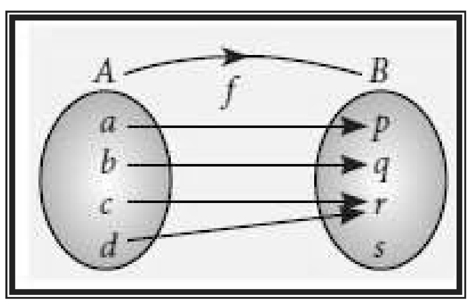
If a function *f* is defined from a set A to set B then (if : A $\rightarrow$ B) set A is called the domain of f and set B is called the co-domian of *f*.

The set of all *f*-images of the elements of A is called the range of *f*.

In other words, we can say

Domain = All possible values of x for which $f(x)$ exists.

Range = For all values of , all possible values of $f(x)$.



From the figure we observe that

Domain = A = {*a, b, c, d*} Range = {*p, q, r*}, Co-Domain = {*p, q, r, s*} = B

EQUAL FUNCTION

Two function *f* and *g* are said to be equal functions, if and only if

(i) Domain of f = Domain of g

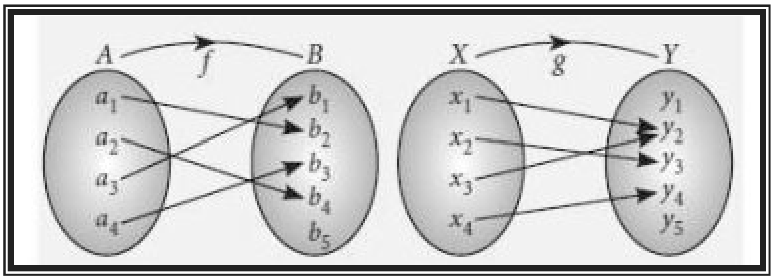
(ii) Co-domain of *f* = Co-domain of g

(iii) $f(x) = g(x)$ for all $x \in$ their common domain

TYPES OF FUNCTION

One-one function (injection): A function $f : A \rightarrow B$ is said to be a one-one function or an injection, if different elements of A have different images in B.

e.g. Let $f : A \rightarrow B$ and $g : X \rightarrow Y$ be two functions represented by the following diagrams.



Clearly, $f : A \rightarrow B$ is a one-one function. But $g : X \rightarrow Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function g.

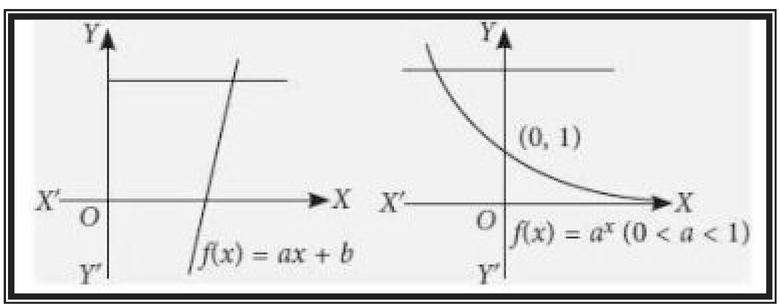
Method to check the injectivity (One-One) of a function

(i) Take two arbitrary elements x, y (say) in the domain of f.

(ii) Solve $f(x) = f(y)$. If $f(x) = f(y)$ give $x = y$ only, then $f : A \rightarrow B$ is a one-one function (or an injection). Otherwise not.

If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

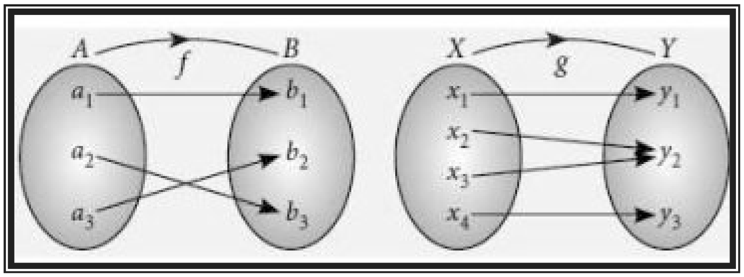
If the graph of the function $y = f(x)$ is given and each line parallel to x-axis cuts the given curve at maximum one point then function is one-one. (Strictly increasing or Strictly Decreasing Function). E.g.



Number of one-one functions (injections) : If A and B are finite sets having m and n elements respectively, then number of one-one functions from A and B = ${}^{n}P_{m}$ is $n \geq m$ and 0 if $n < m$.

If $f(x)$ is not one-one function, then its Many-one function.

Onto function (surjection) : A function $f : A \rightarrow B$ is onto if each element of B has its pre-image in A. In other words, Range of f = Co-domain of f. *e.g.* The following arrow-diagram shows onto function.

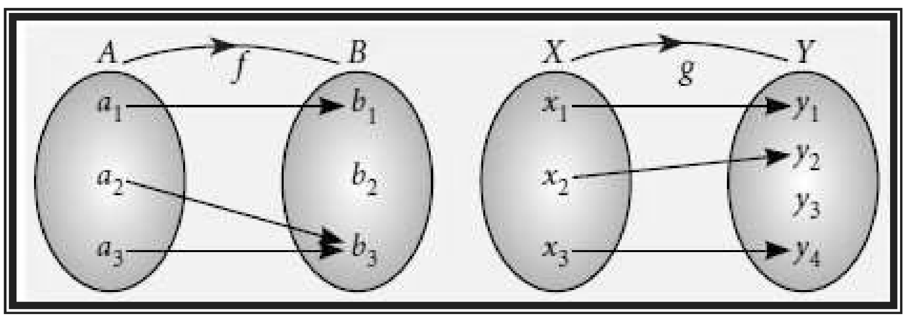


Number of onto function (surjection): If A and B are two sets having m and n elements respectively such that $1 \le n \le m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r}\ {}^{n}C_{r}$

$. r^m$

Into function: A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words, $f : A \to B$ is an into function if it is not an onto function e.g, The following arrow diagram shows into function.



Method to find onto or into function:

(i) Solve $f(x) = y$ by taking x as a function of y i.e., $g(y)$(say).

(i) Now if $g(y)$ is defined for each $y\ \varepsilon$ co-domain and $g(y)\ \varepsilon$ domain then $f(x)$ is onto and if any one of the above requirements is not fulfilled, then *f(x)* is into.

One-one onto function (bijection) : A function $f : A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f : A \to B$ is a bijection if

(i) It is one-one ie., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) It is onto i.e., for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Clearly, f is a bijection since it is both injective as well as surjective.

Illustration :

Let $f : R \to R$ be defined as $f(x) = 7x - 5$, then show that function is one-one and onto Both.

Solution : Let $f(x) = f(y)\ \forall\ x, y \in R$

$\Rightarrow \quad 7x - 5 = 7y - 5$

$\Rightarrow \quad x = y$, so **$f(x)$ is one-one function**

Now, As $f(x) = 7x - 5$, is a polynomial function.

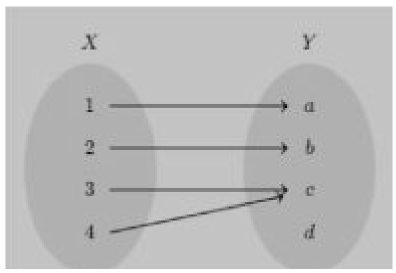
so it is defined everywhere. Thus, Range = R

As, Range = co-domain, so **f is onto function.**

Alternative method : Graph of $f(x)$ is a line which is strictly increasing for all values of x, so its one-one function and Range of $f(x)$ is R which is equal to R so onto funclion.

ILLUSTRATION:

If $f: X \rightarrow Y$ is defined, then show that f is neither one-one nor onto function.



Solution : As for elements 3 and 4 from set X we have same image c in set Y, so f is not one-one function.

Further element d has no pre -image in set X,

so f is not onto function

ILLUSTRATION:

Prove that the function $f: N \rightarrow N$, defined by $f(x) = x^2 + x + 2022$ is one-one.

SOLUTION : APPROACH-I

Let $f(x_1) = f(x_1) \;\forall\; x_1, x_2 \in N \Rightarrow = x_1^2 + x_1 + \cancel{2022} = x_2^2 + x_2 + \cancel{2022}$

$\Rightarrow \quad x_1^2 + x_1 = x_2^2 + x_2$

$\Rightarrow \quad (x_1^2 - x_2^2) + (x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2) \times (x_1 + x_2 + 1) = 0$

Thus, $(x_1 - x_2) = 0$ as $(x_1 + x_2 + 1) \neq 0 \;\forall\; x_1, x_2 \in N \Rightarrow x_1 = x_2$

so, f is ONE-ONE function

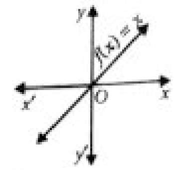
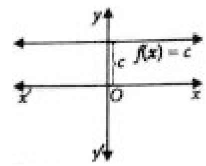
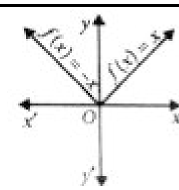
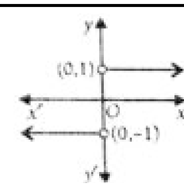
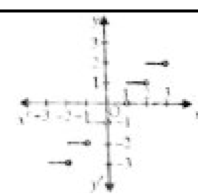
APPROACH-II

$f(x) = x^2 + x + 2022 \Rightarrow f'(x) = 2x + 1$

As, $x \in N$ so, $2x + 1 > 0 \Rightarrow f'(x) = 0$ (Strictly Increasing function)

so, f is ONE-ONE function

Type of Functions

Name of Function	Definition	Domain	Range	Graph
1. Identify Function	The function $f: R \to R$ defined by $f(x) = x \ \forall \ x \in R$	R	R	
2. Constant Function	The function $f: R \to R$ defined by $f(x) = c \ \forall \ x \in R$	R	(c)	
3. Polynomial Function	The function $f: R \to R$ defined by $f(x) = p_0 + p_1x + p_2x^2 + ... + p_nx^n$, where $n \in N$ and $p_0, p_1, p_2 ..., p_n \in R \ \forall \ x \in R$			
4. Rational Function	The function f defined by $f(x) = \frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomial functions, $Q(x) \neq 0$			
5. Modulus Function	The function $f: R \to R$ defined by $f(x) = \|x\| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \ \forall \ x \in R$	R	$[0, \infty)$	
6. Signum Function	The function $f: R \to R$ defined by $f(x) = \begin{cases} \frac{\|x\|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \\ 0, & x = 0 \end{cases}$	R	$\{-1, 0, 1\}$	
7. Greatest Integer Function	The function $f: R \to R$ defined by $f(x) = \|x\| = \begin{cases} x, x \in Z \\ \text{integer less than} \\ \textit{equal to } x, x \notin Z \end{cases}$	R	Z	
8. Linear Function	The function $f: R \to R$ defined by $f(x) = mx + c$, $x \in R$ where m and c are constants	R	R	

ONE–MARK QUESTIONS

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE

1. Consider the set A = {1, 2, 3}, then write smallest equivalence relation on A.

   (a) { } (b) {(1, 1)} (c) {(1, 1), (2, 2), (3, 3)} (d) {(3, 3)}

2. Consider the set A containing 5 elements, then the total number of injective functions from A to itself are

   (a) 5 (b) 25 (c) 120 (d) 125

3. Let Z be the set of integers and R be the relation defined in Z such that *aRb* if $(a - b)$ is divisible by 4, then R partitions the set Z into how many Pairwise disjoint subsets.

   (a) 2 (b) 3 (c) 4 (d) 5

4. If A = {d, 0, e} then the number of relations on A × A are

   (a) 3 (b) 8 (c) 15 (d) 512

5. If A = {2023, 2024) then the number of non-empty relations on A × A are

   (a) 1 (b) 4 (c) 15 (d) 16

6. If A = {2023, 2024) then the number of Reflexive relations on A × A are

   (a) 2 (b) 4 (c) 8 (d) 16

7. If A = {s, u, v}, then the number of Symmetric relations on A × A are

   (a) 8 (b) 9 (c) 32 (d) 64

8. Let A be the set of the Letters of the name of our country the "INDIA". Then find the number of reflexive relations on A × A

   (a) 4096 (b) 2048 (c) 1024 (d) 16

9. Let $A = \{x : x^2 < 3, x \in W\}$, then the number of Symmtric relations on $A \times A$ are

   (a) 1 (b) 2 (c) 4 (d) 8

10. If there are 'p' elements in set A, such that number of Reflexive relation on A × A are 4096, then p =

    (a) 4 (b) 6 (c) 8 (d) 12

11. Let A = {d, 0, e}, then Find 'p' if the number of Symmetric relations on A × A are 2^p.

    (a) 4 (b) 6 (c) 8 (d) 12

12. Find the maximum number of equivalence relations on the set A = {1, 2, 3}.

    (a) 3 (b) 5 (c) 8 (d) 9

13. If the function $f : R — \{1, -1\} \to A$ defined by $f(x) = \frac{x^2}{1+x^2}$ is Surjective, then A =

    (a) R (b) R — {1, – 1} (c) [0, 1) (d) [0, ∞]

14. The number of injections possible from A = {1, 2, 3, 4} to B = {5, 6, 7} are

    (a) 0 (b) 3 (c) 6 (d) 12

15. If the number of one-one functions that can defined from A = {4, 8, 12, 16} to B is 5040, then $n(B)$ =

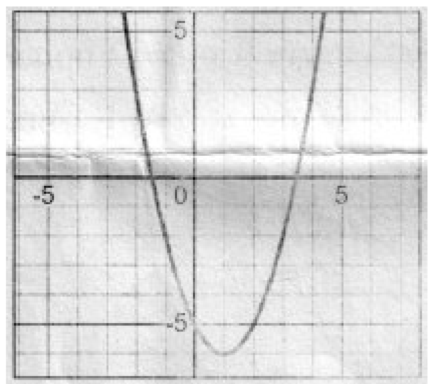
(a) 7 (b) 3 (c) 6 (d) 10

16. If the function $f : R \to A$ defined by $f(x) = 3 \sin x + 4\cos x$ is Surjective, then A =

(a) [–7, 7] (b) [–1, 1] (c) [1, 7] (d) [–5, 5]

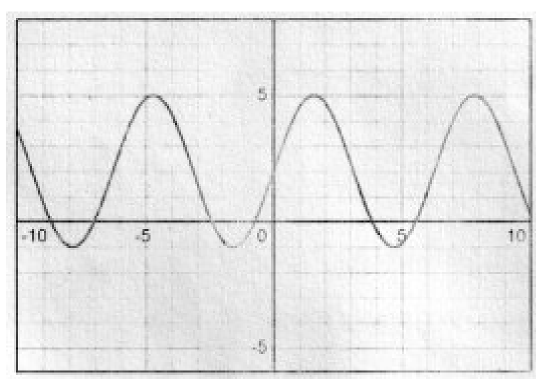
17. The Part of the graph of a Non-Injective function $f : R \to$ Range defined by $f(x) = x^2 - 2x + a$ is given below. If the domain of $f(x)$ is modified as either $(-\infty, b]$ or $[b, \infty]$ then $f(x)$ becomes the Injective function. What must be the value of $(b - a)$.

(a) 6 (b) 5 (c) 4 (d) 0



18. The graph of the function $f : R \to A$ defined by $y = f(x)$ is given below, then find A such that function $f(x)$ is onto function

(a) [–1, 5] (b) [–5, 5] (c) [–5, 1] (d) R



ASSERTION-REASON BASED QUESTIONS (Q.19 & Q.20)

In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are ture but R is not the correct explanation of A.

(c) A is true but R is false

(d) A is false but R is true

19. ASSERTION (A) : A relation R = {(a, b) : |a – b| < 1} defined on the set

A = {1, 2, 3, 4} is Reflexive

Reason (R) : A realtion R on the set A is said to be reflexive if for $(a, b) \in R$ & $(b, c) \in R$, we have $(a, c) \in R$.

20. Assertion (A) : A function $f : R \to R$ given $f(x) = |x|$ is one-one function.

Reason (R) : A function $f : A \to B$ is said to be Injective if

$f(a) = f(b) \Rightarrow a = b$

TWO MARKS QUESTIONS

21. If A = {*a, b, c, d*} and *f* = {(*a, b*), (*b, d*), (*c, a*), (*d, c*)}, show that *f* is one-one from A to A.

22. Show that the relation R on the set of all real numbers defined as R = {(a, b) : $a \leq b^3$} is not transitive.

23. If the function $f : R - \{1, -1\} \to A$ defined by $f(x) = \dfrac{x^2}{1 - x^2}$, is Surjective, then find A.

24. Give an example to show that the union of two equivalence relations on a set A need not be an equivalence relation on A.

25. How many reflexive relations are possible in a set A whose (A) = 4. Also find How many symmetric relations are possible on a set B whose $n(B) = 3$.

26. Let W denote the set of words in the English dictionary. Define the relation R by R={(*x, y*) ε W × W such that *x* and *y* have at least one letter in common}. Show that this relation R is reflexive and symmetric, but not transitive.

27. Show that the relation R in the set of all real numbers, defined as R = {(*a, b*): $a \leq b^2$} is neither reflexive Nor symmetric.

28. Consider a function $f : R_+ \to (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where R+ is the set of all positive real numbers. Show that function is one-one and onto both.

29. Let L be the set of all lines in a plane. A relation R in L is given by R = {(L_1, L_2): L_1 and L_2 intersect at exactly one point, L_1, L_2 ε L}, then show that the relation R is symmetric Only.

30. Show that a relation R on set of Natural numbers is given by R = {(*x, y*): *xy* is a square of an integer} is Transitive.

THREE MARKS QUESTIONS

31. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.

(i) {(*x, y*) : *x* is a person, *y* is the mother of *x*}.

(ii) {(*a, b*): *a* is a person, *b* is an ancestor of *a*}.

32. Show that the function $f: R \to R$ defined by $f(x) = \frac{x^2}{x^2+1}$; $\forall\ x \in R$, is neither one-one nor onto.

33. Let R be the set of real numbers and $f: R \to R$ be the function defined by $f(x) = 4x + 5$. Show that *f* is One-one and onto both.

34. Show that the relation R in the set A = {3, 4, 5, 6, 7} given by R = {(*a, b*) : |a – b| is divisible by 2} is an equivalence relation. Show that all the elements of {3, 5, 7} are related to each other and all the ements of {4, 6} are related to each other, but no element of {3, 5, 7} is related to any element {4, 6}.

35. Check whether the relation R in the set Z of integers defined as R = ((*a*, *b*) : *a* + *b* is "divisible by 2"} is reflexive, symmetric, transitive or Equivalence.

36. Show that that following Relations R are equivalence relation in A.

    (a) Let A be the set of all triangles in a plane and let R be a relation in A, defined by R = {(T_1, T_2) : T_1 is congruent T_2}

    (b) Let A be the set of all triangles in a plane and let R be a relation in A, defined by R = {(T_1, T_2) : T_1 is similar T_2 }

    (c) Let A be the set of all lines in xy-plane and let R be a relation in A, defined by R = {(L_1, L_2,) : L_1 is parallel to L_2}

    (d) Let A be the set of all integers and let R be a relation in A, defined by R = {(*a*, *b*) : (*a* – *b*) is even}

    (e) Let A be the set of all integers and let R be a relation in A, defined by R = {(*a, b*) : |*a* – *b*| is a multiple of 2}

    (f) Let A be the set of all integers and let R be a relation in A, defined by R = {(*a, b*) : |*a* – *b*| is a divisible by 3}

37. Check whether the following Relations are Reflexive, Symmetric or Transitive.

    (a) Let A be the set of all lines in *xy*-plane and let R be a relation in A, defined by R = {(L_1, L_2) : L_1 is perpendicular to L_2}

    (b) Let A be the set of all real numbers and let R be a relation in A defined by R = {(*a, b*): $a \le b$}

    (c) Let A be the set of all real numbers and let R be a relation in A defined by R = {(*a, b*): $a \le b^2$}

    (d) Let A be the set of all real numbers and let R be a relation in A defined by R = {(*a*, *b*) : $a \le b^3$}

    (e) Let A be the set of all natural numbers and let R be a relation in A defined by

R = (a, b) : a is a factor b}

OR

R {(a.b): b is divisible by a}

(f) Let A be the set of all real numbers and let R be a relation in A defined by

R {(a.b): (1+ ab) > 0}

38. Let S be the set of all real numbers. Show that the relation R = {a, b): $a^2 + b^2 = 1$} is symmetric but neither reflexive nor transitive.

39. Check whether relation R defined in R as R = {a, b): $a^2 - 4ab + 3b^2 = 0$, $a, b \in$ R} is reflexive, symmetric and transitive.

40. Show that the function $f : (-\infty, 0) \to (-1, 0)$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in (-\infty, 0)$ is one-one and onto.

FIVE MARKS QUESTIONS

41. For real numbers x and y, define $x R y$ if and only if $x - y + \sqrt{2}$ is an irrational number. Then check the reflexivity, Symmetricity and Transitivity of the relation R.

42. Determine whether the relation R defined on the set of all real numbers as

R = {(a, b) : a, b ε R and a – b + $\sqrt{3}$ ε S}

(Where S is the set of all irrational Numbers) is reflexive, symmetric or transitive.

43. Let N be the set of all natural numbers and let R be a relation on N × N, defined by Show that R is an equivalence relation.

(i) (a, b) R $(c, d) \Leftrightarrow a + d = b + c$

(ii) (a, b) R $(c, d) \Leftrightarrow ad = bc$

(iii) (a, b) R $(c, d) \Leftrightarrow \frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c}$

(iv) (a, b) R $(c, d) \Leftrightarrow ad\,(b + c) = bc\,(a + d)$

44. Let A = R – {1}, $f : A \to A$ is a mapping defined by $f(x) = \frac{x-2}{x-1}$, show that f is one-one and onto.

45. Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \to S$, where S is the range of f, is One-One and Onto Function.

CASE STUDIES

A. A person without family is not complete in this world because family is an integral part of all of us Human deings are considered as the social animals living in group called as family. Family plays many important roles throughout the life.

Mr. D.N. Sharma is an Honest person who is living happily with his family. He has a son Vidya and a Daughter Madhulika. Mr. Vidya has 2 sons Tarun and Gajender and a daughter Suman while Mrs. Madhulika has 2 sons Shashank and Pradeep and 2 daughters Sweety and Anju. They all Lived together and everyone shares equal responsibilities within the family. Every member of the family emotionally attaches to each other in their happiness and sadness. They help each other in their bad times which give the feeling of security.

A family provides love, warmth and security to its all members throughout the life which makes it a complete family. A good and healthy family makes a good society and ultimately a good society involves in making a good country.

On the basis of above information, answer the following questions:

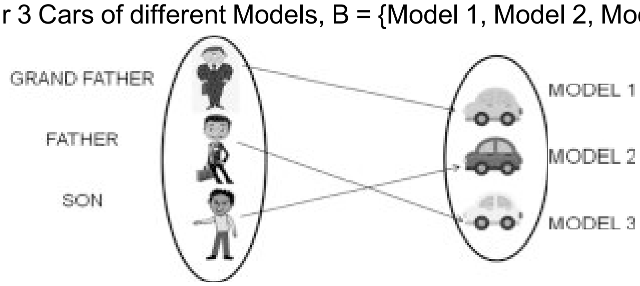
Consider Relation R in the set A of members of Mr. D. N. Sharma and his family at a particular time

(a) If R = {(x, y) : x and y live in the same locality), then R show that R is reflexive Relation.

(b) If R = {(x, y) : x is exactly 7 cm taller than y}, then R show that R is not Symmetric relation.

(c) If R = {(x, y) : x is wife of y}, then show that R is Transitive only.

B. Let A be the Set of Male members of a Family, A = (Grand father, Father, Son) and B be the set of their 3 Cars of different Models, B = {Model 1, Model 2, Model 3}



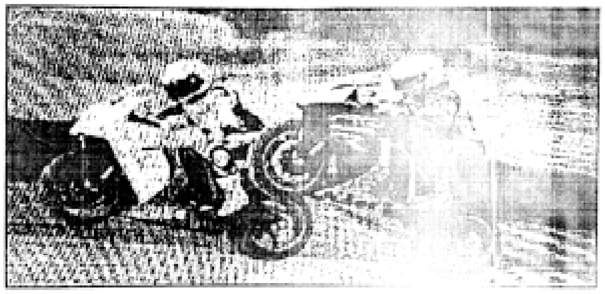
On the basis of The above Information, answer the following questions:

(a) If m & n represents the total number of Relations & functions respectively on A × B, then find the value of (m + n).

(b) If p & q represents the total number of Injective function & total numbers of Surjective functions respectively on A × B, then find the value of $|p - q|$.

C. An organization conducted bike race under two different categories—Boys & Girls. There were 28 participants in all. Among all of them finally three from category 1 and two from category 2 were selected for the final race. Ravi forms two sets B and G with these Participants for his college Project.

Let B = {b1, b2, b3} and G = {g1, g2}, represents the set of Boys selected & G the set of Girls selected for the final race.



(a) How many relations are possible from B to G?

(b) Among all possible relations from B to G, how many functions can be formed from B to G?

(c) Let R:B $\to$ B be defined by

R = {(x, y) : x & y are students of same sex}. Check R is equivalence Relation.

OR

A function f : B $\to$ G be defined by $f = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$

Check if f is bijective. Justify your answer.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE:

1. Consider the set A = {1. 2, 3} and R be the smallest equivalence relation on A.then R =

   (a) {(1.1)} (b) {(1,1), (2,2)}

   (c) {(1,1),(2,2),(3,3)} (d) ϕ

2. Consider the set A containing n elements. Then, the total number of injective functions from A onto itself is

   (a) 2^n (b) n

   (c) n (d) $n!$

3. The total number of injective nappingsfrom a set with m elements to a set with n elements, $m \leq n$ is

   (a) $n!$ (b) n^m

   (c) m^n (d) $\dfrac{n!}{(n-m)!}$

4. The number of injections possible from A = {1,3,5,6} to B = {2,8,11} is

(a) 12 (b) 22

(c) 3 (d) 0

5. The number of one-one functions that can defined from

A = {4,8,12,16} to B is 5040, then n(B) =

(a) 7 (b) 8

(c) 9 (d) 10

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT CHOOSE THE CORRECT ALTERNATIVE.

1. A relation R in a set A is called .... if $(a_1, a_2) \in$ R implies $(a_2, a_1) \in$ R, for all $a_1, a_2 \in$ A.

(a) Reflexive (b) Symmetric

(c) Transitive (d) Equivalence

2. Let f : R – {0} → R – {0} be defined by $f(x) = \frac{1}{x}\ \forall\ x \in$ R. Then f is

(a) One-One (b) Many-One

(c) Not defined (d) None of these

3. Let P = $\{(x, y) \mid x^2 + y^2 = 1, x, y \in R\}$. Then P is

(a) Re flexive (b) Symmetric

(c) Transitive (d) Equivalence

4. The function f : R → R defined by $f(x) = [x]$, where [.] is greatest integer function is

(a) One-One (b) Many -One

(c) Onto (d) None of these

5. The number of bijective functions (One-one and onto both) from set A to itself when A contains 2022 elements is

(a) 2022 (B) 2022!

(C) 2022^2 (D) 2022^{2022}

ANSWER

One Mark Questions

1. (c) {(1,1), (2,2), (3,3)}
2. (c) 120
3. (c) 4
4. (d) 512
5. (c) 15
6. (b) 4
7. (d) 64
8. (a) 4096
9. (d) 8
10. (a) 4
11. (b) 6
12. (b) 5
13. (c) [0,1)
14. (a) 0
15. (d) 10
16. (d) [-5,5]
17. (a) 6
18. (a) [-1,5]
19. (c) A is true but R is false
20. (d) A is false but R is true

Two Mark Questions

23. A = R – [–1, 0)

25. Reflexive Relations = 4096 Symmetric Relation = 64

Three Mark Questions

31. (a) Yes it's function, Not Injective but Surjective (b) No, its not a function

35. EQUIVALENCE RELATION

37. (a) Symmetric (b) Reflexive and Transitive
    (c) Neither Reflexive, Symmetric nor Transitive
    (d) Neither Reflexive, Symmetric nor Transitive
    (e) Reflexive and Transitive
    (f) Reflextive and Symmetric

39. Reflexive only

Four/Five Mark Questions

41. Reflexive only
42. Reflexive only

CASE STUDIES BASED QUESTION

B. (a) 512 +27 =539

B. (b) 0

C. (a) 64
   (b) 8
   (c) R is an Equivalence Relation OR (c) f is not Bijective

SELF ASSESSMENT-1

1. (c) 2. (d) 3. (d) 4. (d) 5. (d)

SELF ASSESSMENT-2

1. (b) 2. (a) 3. (b) 4. (b) 5. (b)

CHAPTER-2

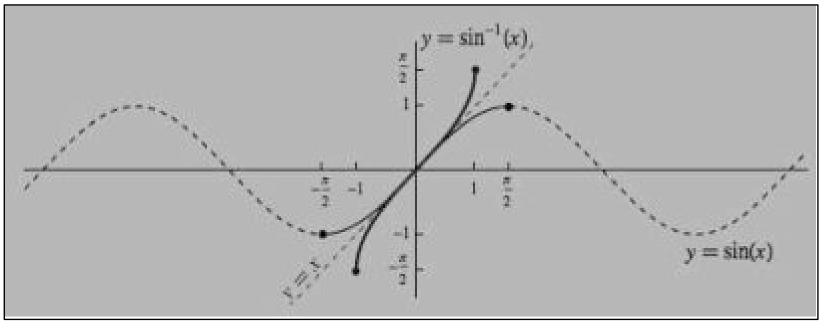
INVERSE TRIGONOMETRIC FUNCTIONS

An example of people using inverse trigonometric functions would be builders such as construction workers, architects, and many others.

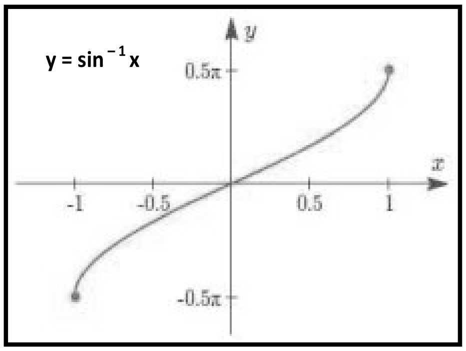
An example of the use would be the creation of bike ramp. You will have to find the height and the length. Then find the angle by using the inverse of sine. Put the ength over the height to find the angle. Architects would have to calculate the angle of a bridge and the supports when drawing outlines. These calculations are then applied to find the safest angle. The workers would then uses these calculations to build the bridge.

TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM (2025-26)

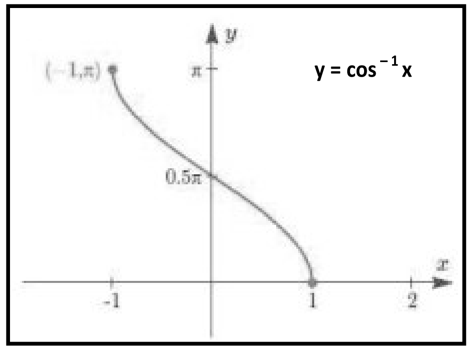
- Definition, range, domain, principal value branch.
- Graphs of inverse trigonometric functions.



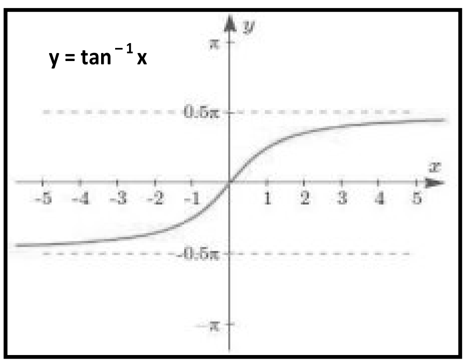
y = sin⁻¹(x)
y = sin(x)
y = x



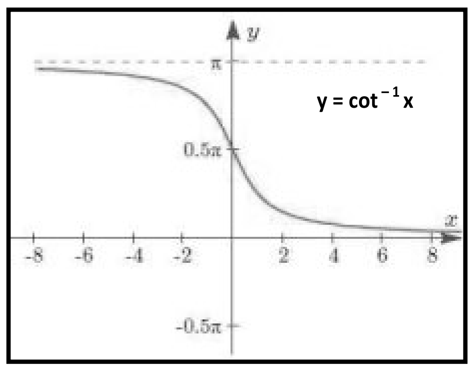
y = sin⁻¹x
0.5π
-0.5π
-1
-0.5
0.5
1



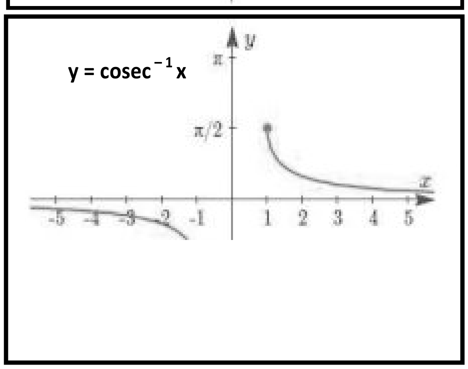
y = cos⁻¹x
(−1, π)
π
0.5π
-1
1
2



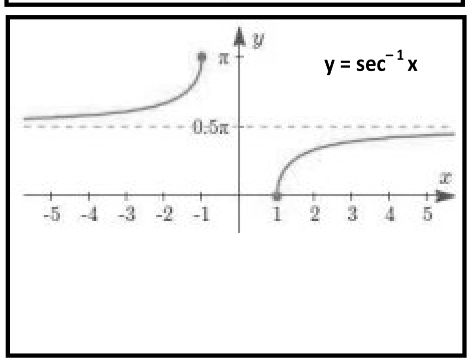
y = tan⁻¹x
π
0.5π
-0.5π
−π



y = cot⁻¹x
π
0.5π
-0.5π



y = cosec⁻¹x
π
π/2



y = sec⁻¹x
π
0.5π

Function	Domain	Range
$y = \sin^{-1} x$	[–1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	[–1, 1]	$[0, \pi]$
$y = \tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$(0, \pi)$
$y = \sec^{-1} x$	R – (–1, 1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \text{cosec}^{-1} x$	R – (–1, 1)	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- when $x \in [-1, 1]$

  $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.
- when $x \in R$

  $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.
- when $x \in R - (-1, 1)$

  $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$.

- $\sin^{-1} (\sin x) = x$, when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos^{-1} (\cos x) = x$, when $x \in [0, \pi]$
- $\tan^{-1} (\tan x) = x$, when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\cot^{-1} (\cot x) = x$, when $x \in (0, \pi)$
- $\text{cosec}^{-1} (\text{cosec } x) = x$, when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- $\sec^{-1} (\sec x) = x$, when $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

- $\sin (\sin^{-1} x) = x$, when $x \in [-1, 1]$
- $\cos (\cos^{-1} x) = x$, when $x \in [-1, 1]$
- $\tan (\tan^{-1} x) = x$, when $x \in R$
- $\cot (\cot^{-1} x) = x$, when $x \in R$
- $\text{cosec} (\text{cosec}^{-1} x) = x$, when $x \in R - (-1, 1)$
- $\sec (\sec^{-1} x) = x$, when $x \in R - (-1, 1)$

- $\sin^{-1}(-x) = -\sin^{-1} x$, when $x \in [-1, 1]$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$, when $x \in [-1, 1]$
- $\tan^{-1}(-x) = -\tan^{-1} x$, when $x \in R$
- $\cot^{-1}(-x) = \pi - \cot^{-1} x$, when $x \in R$
- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, when $x \in R\ (-1, 1)$
- $\sec^{-1}(-x) = \pi - \sec^{-1} x$, when $x \in R\ (-1, 1)$

Illustration:

Find the principal value of $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$.

Solution: As, $\sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$, $\boxed{\frac{\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]}$

$\cos^{-1}\left(\frac{-1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3}$, $\boxed{\frac{\pi}{3} \in [0, \pi]}$

so, $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{6} + \pi - \frac{\pi}{3} = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$

Illustration:

Find the principal value of $\sec^{-1}(2) + \sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3})$.

Solution: As, $\sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right)$

$\tan^{-1}(-\sqrt{3}) = -\tan^{-1}(\sqrt{3}) = -\tan^{-1}\left(\tan\frac{\pi}{3}\right) = -\frac{\pi}{3}$, $\boxed{-\frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$

$\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3}) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

Illustration:

Find the range of the function $f(x) = \tan^{-1} x + \cot^{-1} x$.

Solution: As, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

so, $f(x) = \frac{\pi}{2}$ (A constant function)

Thus range of $f(x)$ is $\left\{\frac{\pi}{2}\right\}$.

Illustration:

If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then find the value of $\cos^{-1} x + \cos^{-1} y$.

Solution: As, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \Rightarrow \boxed{\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x}$

$\cos^{-1} x + \cos^{-1} y = \pi - (\sin^{-1} x + \sin^{-1} y) = \pi - \frac{2\pi}{3} = \boxed{\frac{\pi}{3}}$

Illustration:

If $a \le 2 \sin^{-1} x + \cos^{-1} x \le b$, then find the value a and b.

Solution: We know that, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ and $\frac{-\pi}{2} \le \sin^{-1} x \frac{\pi}{2}$,

$\Rightarrow \quad 0 \le (\sin^{-1} x) + \frac{\pi}{2} \le \pi$

$\Rightarrow \quad 0 \le (\sin^{-1} x) + \sin^{-1} x + \cos^{-1} x \le \pi$

$\Rightarrow \quad 0 \le 2 \sin^{-1} x + \cos^{-1} x \le \pi$, but given, $a \le \sin^{-1}$, $x + \cos^{-1} x \le b$

$\boxed{\text{Thus, } a = 0 \text{ and } b = \pi}$

Illustration:

If $\sin[\cot^{-1}(1 + x)] = \cos[\tan^{-1} x]$, then find x.

Solution: As, $\sin[\cot^{-1}(1 + x)] = \cos[\tan^{-1} x]$

$$\Rightarrow \quad \sin\left[\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right] = \cos\left[\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow \quad x^2 + 2x + 2 = 1 + x^2$$

$$\Rightarrow \quad 2x = -1 \Rightarrow \boxed{x = -0.5}$$

Illustration:

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then prove that $xy + yz + zx = 1$.

Solution: Let, $\tan^{-1} x = A$, $\tan^{-1} y = B$, $\tan^{-1} z = C$

so, $A + B + C = \frac{\pi}{2} \Rightarrow A + B = \frac{\pi}{2} - C$

$$\tan(A + B) = \tan\left(\frac{\pi}{2} - C\right) = \cot C$$

$$\frac{\tan A + \tan B}{1 - \tan A.\tan B} = \frac{1}{\tan C} \Rightarrow \frac{x+y}{1-xy} = \frac{1}{z}$$

$$\Rightarrow \quad xz + yz = 1 - xy$$

$$\Rightarrow \quad \boxed{xz + yz + zx = 1}$$

ONE MARK QUESTIONS

1. Principal Value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$ *is*

   (a) π (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{6}$

2. Principal Value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is

(a) $\frac{3\pi}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{\pi}{2}$ (d) $\frac{-3\pi}{5}$

3. Principal value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is

(a) $\frac{4\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) π (d) $\frac{14\pi}{3}$

4. If the Principal value of $\tan^{-1}(\tan\frac{7\pi}{6})$ is $\frac{a\pi}{b}$, Where *a* & *b* are co-prime numbers, then (*a* + *b*)=

(a) 13 (b) –13 (c) 7 (d) 5

5. If the Principal value of $\cos^{-1}(\cos\frac{2\pi}{3}) + \sin^{-1}(\sin\frac{2\pi}{3})$ is $\frac{a\pi}{b}$, then $|a - b|$=

(a) 0 (b) 1 (c) 2 (d) 4

6. If $\cos(\cos^{-1}\frac{1}{3} + \sin^{-1}x) = 0$, then (3x +1)=

(a) 0 (b) 1 (c) 2 (d) 4

7. If $\sin(\sin^{-1}\frac{3}{5} + \cos^{-1}x) = 1$, then (5x -2)=

(a) 0 (b) 1 (c) 2 (d) 4

8. Domain of the function $\cos^{-1}(2x - 1)$ is

(a) *R* (b) [–1,1] (c) [0,1] (d) [0,2]

9. Domain of the function $f(x) = \sin^{-1}\sqrt{x-1}$ is

(a) [1,2] (b) [–1,1] (c) [0,1] (d) [0,2]

10. Principal value of $\sec^{-1}(2) + \sin^{-1}(\frac{1}{2}) + \tan^{-1}(-\sqrt{3})$ is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

11. Domain of the function $f(x) = \cos^{-1}\sqrt{x+1}$ is

(a) [1,2] (b) [–1,0] (c) [0,1] (d) [0,2]

12. Domain of the function f (x) = $\sin^{-1}(-x^2)$ is

(a) [1,2] (b) [–1,0] (c) [0,1] (d) [–1,1]

13. Domain of the function f(x) = $\sin^{-1}(2x+3)$ is

(a) [–2,2] (b) [–2,–1] (c) [0,1] (d) [–1,1]

14. If Domain of the function $f(x) = \sin^{-1}(x^2 - 4)$ is $[-b, -a] \cup [a, b]$ then the value of $(a^2 + b^2)$ is.

(a) 8 (b) 3 (c) 5 (d) 4

15. If $\sin^{-1} x_1 + \sin^{-1} x_2 = \pi$, then the value of $(x_1 + x_2)$ is

(a) 0 (b) 1 (c) –1 (d) 2

16. If $\cos^{-1} a + \cos^{-1} b = 2\pi$, then the value of $(a - b)^2$ is

(a) 0 (b) 1 (c) –1 (d) 4

17. $\cos^{-1}[\sin(\cos^{-1}\frac{1}{2})]=$

(a) $\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

18. Principal value of $\sin^{-1}(\cos\frac{34\pi}{5})$ is

(a) $\frac{\pi}{5}$ (b) $\frac{-\pi}{10}$ (c) $\frac{3\pi}{10}$ (d) $\frac{-3\pi}{10}$

19. If $\cot(\cos^{-1}\frac{7}{25}) = x$, then $\sqrt{24x+2} =$

(a) 1 (b) 2 (c) 3 (d) 4

20. If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$ & $\cot^{-1} x + \cot^{-1} y = \frac{k\pi}{5}$, then k =

(a) 1 (b) 2 (c) 3 (d) 4

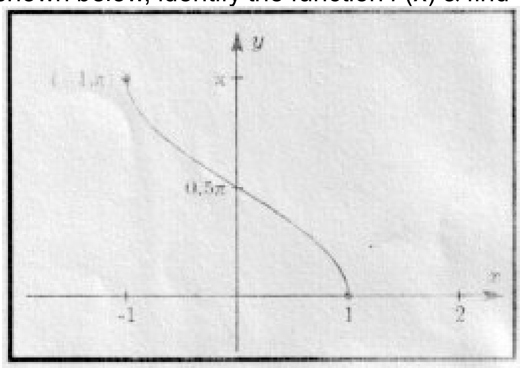
21. $\sum_{i=1}^{2023} \cos^{-1} x_1 = 0$, then the velue of $\sum_{i=1}^{2023} x_i$ is

(a) 0 (b) 1 (c) 2023 (d) –2023

22. If $\sum_{i=1}^{2024} \sin^{-1} x_i = 1012\pi$, then the value of $\sum_{i=1}^{2024} X_i$ is

(a) 1012 (b) 2024 (c) –1012 (d) –2024

23. If graph of f (x) is shown below, identify the function f (x) & find the value of $f\left(-\frac{1}{2}\right)$.



(a) $\frac{-\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{-\pi}{3}$ (d) $\frac{2\pi}{3}$

ASSERTION-REASON BASED QUESTIONS (Q.24 & Q.25)

In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true.

24. ASSERTION (A): The range of the function f (x) = $\sin^{-1}x + \frac{3\pi}{2}$, where

$$x \in [-1,1], \text{ is } [\pi, 2\pi].$$

REASON (R): The range of the principal value branch of $\sin^{-1}x$ is [0, π].

25. ASSERTION (A): All trigonometric function have their inverses over their respective domains.

REASON (R): The inverse of $\tan^{-1}x$ exists for some $x \in R$.

TWO MARKS QUESTIONS

26. Match the following:

If $\cos^{-1}a + \cos^{-1}b = 2\pi$ and $\sin^{-1}c + \sin^{-1}d = \pi$ then

	Column 1		Column 2
A	abcd	P	0
B	$a^2 + b^2 + c^2 + d^2$	Q	1
C	$(d - a) + (c - d)$	R	2
D	$a^3 + b^3 + c^3 + d^3$	S	4

27. Find the value of $\cos\left[\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\sin\frac{5\pi}{3}\right)\right]$

28. If $P = \tan^2(\sec^{-1} 2) + \cot^2(\operatorname{cosec}^{-1} 3)$, then find the value of $(P^2 + P + 11)$.

29. If $P = \sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$, then find the value of $(P^2 - 2P)$.

30. Find the value of $\sin\left(\frac{1}{2}\cot^{-1}\left(\frac{3}{4}\right)\right)$. $\left[\text{Hint} : \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}\right]$

31. Solve for x : $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$

32. Find the value of x, such that $\sin^{-1}x = \frac{\pi}{6} + \cos^{-1}x$.

33. Find x, if $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{2}$

34. If $\tan^{-1}(\cot x) = 2x$, find x.

35. Solve for x : $\cos^{-1}\left(\cos\frac{3\pi}{4}\right) + \sin^{-1}\left(\sin\frac{3\pi}{4}\right) = x$

THREE MARKS QUESTIONS

36. Find the value of k, if 100 $\sin(2\tan^{-1}(0.75)) = k$ [**Hint:** $\sin 2\theta = 2\sin\theta\cos\theta$]

37. Prove that:

(a) $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\dfrac{x}{2},\ x\in\left(0,\ \dfrac{\pi}{4}\right)$

(b) $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)=\dfrac{x}{4}-\dfrac{1}{2}\cos^{-1}x$

(c) $\tan^{-1}\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\dfrac{4-\sqrt{7}}{3}$

(d) $\sin^{-1}\left(2\tan^{-1}\left(\dfrac{2}{3}\right)\right)=\dfrac{12}{13}$

38. (a) Prove that $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]=\sqrt{\dfrac{x^2+1}{x^2+2}}$

(b) Prove that $\tan\left(\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)=\dfrac{2b}{a}$

(c) Prove that $\tan\left(\dfrac{\pi}{4}+\dfrac{1}{2}\tan^{-1}\dfrac{a}{b}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\tan^{-1}\dfrac{a}{b}\right)=\dfrac{2\sqrt{a^2+b^2}}{b}.$

(d) Prove that : $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x$

(e) Prove that : $\tan^{-1}\left(\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)=\dfrac{\pi}{4}+\dfrac{x}{2},\ x\in\left(0,\dfrac{\pi}{2}\right)$

(f) Prove that : $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\dfrac{x}{2},\ x\in\left(0,\dfrac{\pi}{4}\right)$

39. Solve for x :

(a) $\sin^{-1}(6x)+\sin^{-1}(6\sqrt{3}x)=\dfrac{\pi}{2}$

(b) Solve for x : $\sin^{-1}(6x)+\sin^{-1}(6\sqrt{3}x)=\dfrac{-\pi}{2}$

(c) $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\dfrac{5\pi^2}{8}$.

40. Solve for x : $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\dfrac{3}{4}\right),\ x>0$

FIVE MARKS QUESTIONS

Illustration: (For Solving Q.41)

If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then prove that $x^2+y^2+z^2+2xyz=1$.

Solution: Let, $\cos^{-1}x=A$, $\cos^{-1}y=B$, $\cos^{-1}z=C$

so, $A+B+C=\pi \Rightarrow A+B=\pi-C$

Thus, $\cos(A+B)=\cos(\pi-C)$

$\Rightarrow \cos A\cos B-\sin A\sin B=-\cos C$

$\Rightarrow \cos A\cos B-\sqrt{1-\cos^2 A}\sqrt{1-\cos^2 B}=-\cos C$

$\Rightarrow xy-\sqrt{1-x^2}\sqrt{1-y^2}=-z$

$\Rightarrow (xy+z)=\sqrt{1-x^2}\sqrt{1-y^2}$

On squaring both the sides, we get

$(xy+z)^2=(1-x^2)(1-y^2)$

$\Rightarrow \cancel{x^2y^2}+z^2+2xyz=1-x^2-y^2+\cancel{x^2y^2}$

$\therefore \boxed{x^2+y^2+z^2+2xyz=1}$

41. Prove the following:

(a) If $\cos^{-1}\left(\dfrac{x}{a}\right)+\cos^{-1}\left(\dfrac{y}{b}\right)=\alpha$, then prove that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{2xy}{ab}\cos\alpha=\sin^2\alpha$

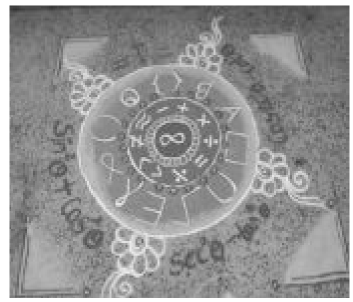
(b) If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta,$ then prove that $9x^2 + 4y^2 - 12xy \cos\theta = 36 \sin^2\theta$.

42. Prove the following:

(a) If $\tan^{-1} x+\tan^{-1} y+\tan^{-1} z=\pi,$ then prove that $x + y + z = xyz$

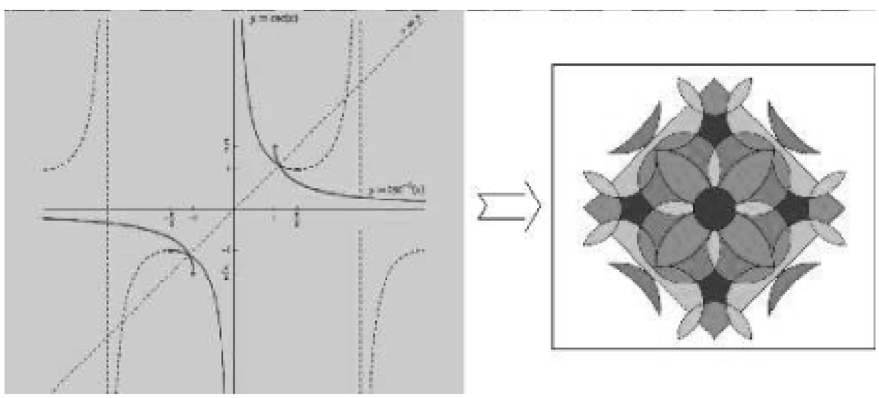
(a) If $\cot^{-1} x+\cot^{-1} y+\cot^{-1} z=\pi,$ then prove that $xy + yz + zx = 1$

CASE STUDIES



43. On National Mathematics Day, December 22, 2020, Mathematics Teachers of DOE organized Mathematical Rangoli Competition for the students of all DOE schools to celebrate and remembering the contribution of Srinivasa Ramanujan to the field of mathematics. The legendary Indian mathematician who was born on this date in 1887.

Team A of class XI students made a beautiful Rangoli on Trigonometric Identities as shown in the figure Above, While Team B of class XII students make the Rangoli on the graph of Trigonometric and Inverse Trigonometric Functions. As shown in the following figure.



On the basis of above information, Teacher asked few questions from Team B. Now you try to answer. Those questions which are as follows:

(a) Write the domain & range (principal value branch) of the function f (x) = $\tan^{-1}$ x?

(b) If the principal branch of $\sec^{-1}x$ is $[0,\pi]-\{k\pi\}$, then find the value of k.

(c) Draw the graph of $\sin^{-1}$ x, where $x \in [-1,1]$. Also write its Principal branch Range.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $\cos\left(\cos^{-1}\frac{2}{3}+\sin^{-1} x\right)=0,$ then (3x – 1)

(a) 0 (b) 1

(c) –1 (d) 2

2. Domain of the function $\cos^{-1}\left(\frac{x}{2}-1\right)$ is

(a) [0, 2] (b) [–1, 1]

(c) [0, 1] (d) [0, 4]

3. If $\cos^{-1}a + \cos^{-1}b = 2\pi$ and $\sin^{-1}c + \sin^{-1}d = \pi$, then $a^2 + b^2 + c^2 + d^2 =$

(a) 0 (b) 1

(c) 2 (d) 4

4. The principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

(a) 0 (b) π

(c) 2π (d) $\frac{4\pi}{3}$

5. If $\cos^{-1}\left(\frac{1}{x}\right) = \theta$, then $\tan\theta =$

(a) x (b) $x^2 + 1$

(c) $\sqrt{x^2+1}$ (d) $\sqrt{x^2-1}$

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then $(x^3 + y^3 + z^3 - 3xyz) =$

(a) 0 (b) 1

(c) –1 (d) 2

2. Principal Range of the function $\sin^{-1}x$ is

(a) $[0, \pi]$ (b) $(0, \pi)$

(c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

3. If $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)=x$, then $x =$

(a) 0 (b) π

(c) $\frac{5\pi}{3}$ (d) $\frac{10\pi}{3}$

4. If $\sin^{-1}\left(\frac{x}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then x =

(a) 0 (b) 1

(c) 2 (d) 3

5. Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is

(a) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (B) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

(C) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (D) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$

ANSWER

One Mark Questions

1. (*d*) $\frac{5\pi}{6}$
2. (*b*) $\frac{2\pi}{5}$
3. (*b*) $\frac{2\pi}{3}$
4. (*c*) 7
5. (*a*) 0
6. (c) 2
7. (*b*) 1
8. (*c*) [0, 1]
9. (a) [1,2]
10. (*d*) $\frac{\pi}{6}$
11. (*b*) [–1,0]
12. (d) [-1,1]
13. (*b*) [–2,–1]
14. (*a*) 8
15. (*d*) 2
16. (*a*) 0
17. (*a*) $\frac{\pi}{6}$
18. (*d*) $\frac{-3\pi}{10}$
19. (*c*) 3
20. (*a*) 1
21. (c) 2023
22. (*b*) 2024
23. (*d*) $\frac{2\pi}{3}$
24. (c) A is true but R is false.
25. (d) A is false but R is true.

Two Marks Questions

26. $A \to Q$, $B \to S$, $C \to R$, $D \to P$
27. 1
28. $(P^2 + P + 11) = 143$
29. $(P^2 - 2P) = 195$
30. $\frac{1}{\sqrt{5}}$
31. $x = 0$ or -1
32. $\frac{\sqrt{3}}{2}$
33. 1

34. $\frac{\pi}{6}$

35. π

Three Marks Questions

36. 96

39. (a) $x = \frac{1}{12}$ (b) $x = \frac{-1}{12}$ (c) $x = -1$

40. $x = \frac{3}{4}$

CASE STUDIES BASED QUESTION

43. (a) Domain $= R = (-\infty, \infty)$, Range $= (\frac{-\pi}{2}, \frac{\pi}{2})$ (b) k = 0.5

SELF ASSESSMENT-1

1. (b) 2. (d) 3. (d) 4. (b) 5. (d)

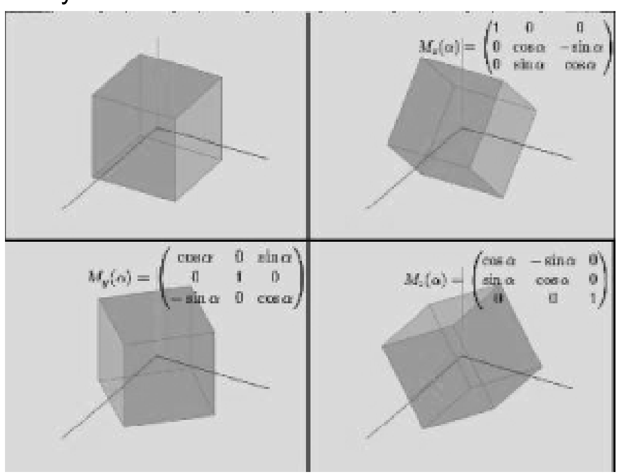
SELF ASSESSMENT-2

1. (a) 2. (c) 3. (a) 4. (d) 5. (c)

CHAPTER-3

MATRICES

Matrices find many applications is scientific field and apply to practical real life problem. Matrices can be solved physical related application and one applied in the study of electrical circuits, quantum mechanics and optics, with the help of matrices, calculation of battery power outputs, resistor conversion of electrical energy into another useful energy, these matrices play a role in calculation, with the help of matrices problem related to Kirchhoff law of voltage and current can be easily solved.



Matrices can play a vital role in the projection of three dimensional images into two dimensional screens, creating the realistic decreeing motion. Now day's matrices are used in the ranking of web pages in the Google search. It can also be used in generalization of analytical motion like experimental and derivatives to their high dimensional.

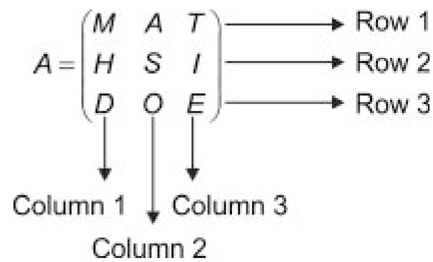
Matrices are also used in geology for seismic survey and it is also used for plotting graphs. Matrices are also used in robotics and automation in terms of base elements for the robot movements. The movements of the robots are programmed with the calculation of matrices 'row and column' controlling of matrices are done by calculation of matrices.

TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM (2025-26)

- Concept, notaion, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices.
- Operation on matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. non commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2).
- Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

Matrices are defined as a rectangular arrangement of numbers of functions. Since it is a rectangular arrangement, it is 2-dimensional.

A two-dimensional matrix consists of the number of rows (m) and a number of columns (n). Horizontal ones are called Rows and Vertical ones are called columns.



ORDER OF MATRIX

The order of matrix is a relationship with the number of elements present in a matrix.

The order of a matrix is denoted by $m \times n$, where m and n are the number of Rows and Columns Respectively and the number of elements in a matrix will be equal to the product of m and n.

TYPES OF MATRICES

Row Matrix

A matrix having only one row is called a row matrix.

Thus $A = [A_{ij}]_{mxn}$ is a row matrix if m = 1. So, a row matirx can be represented as $A = [A_{ij}]_{1xn}$

It is called so because it has only one row and the order of a row matrix will hence be $1 \times n$.

For example,

$A = [1 \quad 2 \quad 3 \quad 4]$ is row matrix of order 1 × 4. Another example of the row matrix is

$B = [0 \quad 9 \quad 4]$ which is of the order 1 × 3.

Column Matrix

A matrix having only one column is called a column matrix. Thus, $A = [A_{ij}]_{mxn}$ is a column matrix if $n = 1$.

Hence, the order is $m \times 1$. An example of a column matrix is:

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, B = \begin{pmatrix} M \\ A \\ T \\ H \end{pmatrix}$$

In the above example, A and B are 3 × 1 and 4 × 1 order matrices respectively.

Square Matrix

If the number of rows and the number of columns in a matrix are equal, then it is called a square matrix.

Thus, $A = [A_{ij}]_{mxn}$ is a square matrix if $m = n$; For example is a square matrix of order 3 × 3.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

<u>For Additional Knowledge:</u>

The sum of the diagonal elements in a square matrix A is called the trace of matrix A, and which is sdenoted by tr(A);

$\text{tr}(A) = a_{11} + a_{22} + ... + a_{nn}$

Zero or Null Matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generlly denoted by O. Thus, $A = [A_{ij}]_{mxn}$ is a zero-matrix if $a_{ij} = 0$ for all i and j; For example

$$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Here A and B are Null matrix of order 3 × 1 and 2 × 2 respectively.

Diagonal Matrix

If all the non-diagonal elements of a square matrix, are zero, then it is called a diagonal matrix.

Thus, a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$;

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A, B and C are diagonal matrix of order 3 × 3, and D is a diagonal matrix of order 2 × 2.

Diagonal matrix can also be denoted by A = diagonal [2 3 4], B = diag [2 0 4], C = [0 0 4]

<u>Important things to note:</u>

(i) A diagonal matrix is always a square matrix.

(ii) The diagonal elements are characterized by this general form: a_{ij}, where $i = j$. This means that a matrix can have only one diagonal.

Scalar Matrix

If all the elements in the diagonal of a diagonal matrix are equal, it is called a scalar matrix.

Thus, a square matrix $A = [a_{ij}]$ is a scalar matrix if

$$A = [a_{ij}] = \begin{cases} 0; & i \neq j \\ k; & i = j \end{cases} \quad \text{Where, } k \text{ is constant.}$$

For example A and B are scalar matrix of order 3 × 3 and 2 × 2 respectively.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix}$$

Unit Matrix or Identity Matrix

If all the elements of a principal diagonal in a diagonal matrix are 1, then it is called a unit matrix.

A unit matrix of order n is denoted by I_n. Thus, a square matrix $A = [a_{ij}]_{m \times m}$ is an identity matrix if

$$A = [a_{ij}] = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases}$$

For example I_3 and I_2 are identity matrix of order 3 × 3 and 2 × 2 respectively.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- All identity matrices are scalar matrices
- All scalar matrices are diagonal matrices
- All diagonal matrices are square matrices

Triangular Matrix

A square matrix is said to be a triangular matrix if the elements above or below the principal diagonal are zero. There are two types of Triangular Matrix:

Upper Triangular Matrix

A square matrix $[a_{ij}]$ is called an upper triangular matrix, if $a_{ij} = 0$, when $i > j$.

$$A = \begin{pmatrix} D & O & E \\ 0 & D & O \\ 0 & 0 & E \end{pmatrix} \text{ is an upper triangular matrix of order } 3 \times 3.$$

Lower Triangular Matrix

A square matrix is called a lower triangular matrix, if $a_{ij} = 0$, when $i > j$.

$$A = \begin{pmatrix} D & 0 & 0 \\ O & D & 0 \\ E & O & E \end{pmatrix}$$

is a lower triangular matrix of order 3 × 3.

Transpose of a Matrix

Let A be any matrix, then on interchanging rows and columns of A. The new matrix so obtained is transpose of A donated A^T or A'.

[order of $A = m \times n$, then order of $A^T = n \times m$]

Properties of transpose matrices A and B are:

(a) $(A^T)^T = A$

(b) $(kA)^T = kA^T$ (k = constant)

(c) $(A + B)^T = A^T + B^T$

(d) $(AB)^T = B^T. A^T$

<u>Symmetric Matrix and Skew-Symmetric matrix</u>

- A square matrix $A = [a_{ij}]$ is symmetric if $A^T = A$ i.e. $a_{ij} = a_{ji} \ \forall \ i$ and j
- A square matrix $A = [a_{ij}]$ is skew-symmetric if $A^T = -A$ i.e. $a_{ji} = -a_{ij} \ \forall \ i$ and j

  (All diagonal elements are zero in skew-symmetric matrix)

Illustration:

A is matrix of order 2022 × 2023 and B is a matrix such that AB^T and B^TA are both defined, then find the order of matrix B.

Solution: Let the order of matrix be R × C, So,

$(A)_{2022\times2023} (B^T)_{C\times R} \quad \Rightarrow \quad C = 2023$ (As AB^T is defined)

$(B^T)_{C\times R} (A)_{2022\times2023} \quad \Rightarrow \quad R = 2022$ (As $B^T A$ is defined)

Thus order of matrix B is (2022 × 2023).

Illustration:

If A is a skew symmetric matrix, then show that A^2 is symmetric.

Solution: As A is skew-symmetric, $A^T = -A$

$(A^2)^T = (A.A)^T = A^T.A^T = (-A)(-A) = A^2$

As $(A^2)^T = A^2$

$\Rightarrow$ Thus, A^2 is symmetric.

Illustration:

If $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} + X = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$, where $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then find the value of $a + c - b - d$.

Solution: As, $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$,

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3-1 & 4+1 \\ 5-2 & 6-3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}$$

On compairing the corresponding elements, we get,

$a = 2$, $b = 5$, $c = 3$, $d = 3$

Thus, $a + c - b - d = 5 - 5 - 3 = -3$

Illustration:

If A is a diagonal matrix of order 3×3 such that $A^2 = A$, then find number of possible matrices A.

Solution: As, A is a diagonal matrix of order 3×3

Let, $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

$$\Rightarrow \quad A^2 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

As $A^2 = A \quad \Rightarrow \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$

So, $a = 0$ or -1, similarly b and c can take 2 values (0 and -1).

Thus, total number of possible matrices are $2 \times 2 \times 2 = 8$.

ONE MARK QUESTIONS

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $A=[a_{ij}]_{2\times2} = \begin{cases} 0, when\, i = j \\ 1, when\, i \neq j \end{cases}$, then $A^2 =$

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

2. If $A=[a_{ij}]_{2\times2} = \begin{cases} 0, when\, i = j \\ 1, when\, i \neq j \end{cases}$, then $A^{2025} =$

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

3. If $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, B = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}$ and $A = B^2$, then x equals

(a) ±1 (b) 1 (c) –1 (d) 2

4. If $A = \begin{pmatrix} 1 & x^2-2 & 3 \\ 7 & 5 & 7 \\ 3 & 7 & -5 \end{pmatrix}$ be a symmetric matrix, then x equals

(a) ±3 (b) ±2 (c) $\pm\sqrt{2}$ (d) 0

5. If $A = \begin{pmatrix} 0 & x^2+6 & 1 \\ -5x & x^2-9 & 7 \\ -1 & -7 & 0 \end{pmatrix}$ be a skew -symmetric matrix, then x equals

(a) ±3 (b) 3 (c) –3 (d) 0

6. If $A = \begin{pmatrix} 2y-7 & 0 & 0 \\ 0 & x-3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ be a scalar matrix, then (x+y) equals

(a) 7 (b) 14 (c) 16 (d) 17

7. If A is matrix of order 2023 × 2024 and B is a matrix such that AB' and B'A both are defined, then the order of matrix B is

(a) 2023 × 2024 (b) 2023 × 2023 (c) 2024 × 2024 (d) 2024 × 2023

8. If A is matrix of order 2023 × 2024 and B is a matrix such that AB and BA both are defined, then the order of matrix B is

(a) 2023 × 2024 (b) 2023 × 2023 (c) 2024 × 2024 (d) 2024 × 2023

9. *If* $A=\begin{pmatrix} 2 & 0 & y-x \\ x+y-2 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ be a diagonal matrix then (xy) equals

(a 1 (b) 2 (c) 3 (d) 4

10. If all entries of a square matrix of order 2 are either 3, –3 *or* 0, then how many Non-zero matrices are possible?

(a) 80 (b) 81 (c) 27 (d) 64

11. If all entries of a square matrix of order 3 are either 1 *or* 0, then how many Diagonal matrices are possible?

(a) 512 (b) 8 (b) 6 (d) 2

12. If all entries of a square matrix of order 3 are either 3 *or* 0, then how many Scalar matrices are possible?

(a) 1 (b) 8 (c) 6 (d) 2

13. If all entries of a square matrix of order 3 are either 5 *or* 0, then how many Identity matrices are possible?

(a) 1 (b) 8 (c) 2 (d) 0

14. If there are five one' s i.e. 1, 1, 1, 1, 1 & four zeroes i.e. 0, 0, 0, 0, then total number of symmetric matrices of order 3 ×3 possible?

(a) 10 (b) 12 (c) 3 (d) 9

15. If $x\begin{pmatrix} 1 \\ 2 \end{pmatrix}+y\begin{pmatrix} 2 \\ 5 \end{pmatrix}=\begin{pmatrix} 4 \\ 9 \end{pmatrix}$, then

(a) x= 1, y = 2 (b) x= 2, y=1 (c) x= 1, y=–1 (d) x= 3, y=2

16. The product $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, is equal to

(a) $\begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$ (b) $\begin{pmatrix} a^2+b^2 & 0 \\ a^2+b^2 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} a^2+b^2 & 0 \\ 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

17. If A is a square matrix such that $A^2 = I$, then $(A - I)^3.+ (A + I)^3 - 7A$ is equal to

(a) I (b) A (c) $3A - I$ (d) $A - I$

18. If A and B are two non-zero matrices such that AB = A, BA = B and $(A + B)^3 = k\,(A+ B)$, then k is equal to

(a) 1 (b) 2 (c) 4 (d) 8

19. If A is a square matrix such that $A^2 = A$, then$(A + I)^2 - 3A$ is equal to

(a) I (b) A (c) $2A$ (d) $3I$

20. If a matrix A = (1 2 3), then the matrix A.A' (where A' is the transpose of A) is

(a) $(1\ 2\ 3)_{1\times 3}$ (b) $(14)_{1\times 1}$ (c) $(6)_{1\times 1}$ (d) $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}_{3\times 1}$

21. If $A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$ and $2A + B$ is a null matrix, then B is equal to]

(a) $\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & -4 \\ -5 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} -6 & -8 \\ -10 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$

22. If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(3I + 4A)\,(3I - 4A) = x^2\,\text{I}$, then value of x is/are

(a) ± 3 (b) $\pm\sqrt{7}$ (c) ± 5 (d) 0

23. If $A = \begin{pmatrix} 2 & 8 \\ 6 & 4 \end{pmatrix} = P + Q,$ where P is a symmetric and Q is a skew-symmetric matrix, then Q is equal to

(a) $\begin{pmatrix} 2 & 6 \\ 8 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

ASSERTION-REASON BASED QUESTIONS (Q.24 & Q.25)

In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true.

24. *ASSERTION:* Matrix A $= \begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & 3 \\ 2 & -3 & 0 \end{pmatrix}$ is a skew-symmetric matrix.

REASONING: A matrix A is skew-symmetric if $A' = -A$.

25. *ASSERTION :* For matrices $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 4 & 2 \\ 9 & 1 \end{pmatrix}$,

$$(A + B)(A - B) = A^2 - AB + BA - B^2$$

REASONING : Matrix multiplication is not commutative.

TWO MARKS QUESTIONS

26. If A is a square matrix, then show that

(a) $(A + A^T)$ is symmetric matrix.

(b) $(A - A^T)$ is symmetric matrix.

(c) (AA^T) is symmetric matrix.

27. Show that every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

28. If A and B are two symmetric matrices of same order, then show that

(i) $(AB - BA)$ is skew-symmetric Matrix.

(ii) $(AB + BA)$ is symmetric Matrix.

29. (a) If $A = \begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$. Verify that $(A + B)C = AC + BC$.

(b) If $A + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $A - 2B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ then show that $A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

30. If $A = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, show that $AB \neq BA$

31. Find a matrix X, for which $\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

32. If A and B are symmetric matrices, show that AB is symmetric, if $AB = BA$.

33. Match the following:

Possible Number of Matrices (A_n) of order 3 × 3 with entry 0 or 1 which are

	Condition		**No. of matrices**
(1)	A_n is diagonal Matrix	P	2^0
(2)	A_n is upper triangular Matrix	Q	2^1
(3)	A_n is identity Matrix	R	2^3
(4)	A_n is scalar Matrix	S	2^6

34. If $A = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$ then prove that $A^3 = \begin{pmatrix} \cos 3x & -\sin 3x \\ \sin 3x & \cos 3x \end{pmatrix}$.

35. Express the following Matrices as a sum of a symmetric and skey-symmetric matrix. (Note: Part (b) and (c) can be asked for one marker, SO THINK ABOUT THIS!)

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 7 \\ 5 & 7 & -5 \end{pmatrix}$ (c) $A = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$

36. Show that the Matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ satisfies the equation $A^2 - 4A + \mathbf{I} = 0$.

37. Find the values of x and y, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfies the equation $A^2 + xA + yI = 0$.

38. Find $f(A)$, if $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$ such that $f(x) = x^2 - 3x + 5$

39. Find A^2 if $A = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$.

40. Find $2A^2$ when $x = \frac{\pi}{3}$ where $A = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$.

THREE MARKS QUESTIONS

41. Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{pmatrix}$ and $Q = [q_{ij}]$ be two 3 × 3 matrices such that $Q = P^5 + I_3$, then Prove that $\left(\frac{q_{21} + q_{31}}{q_{32}}\right) = 10$.

42. Construct a 3 × 3 matrix $A = [a_{ij}]$ such that

(a) $a_{ij} = \begin{cases} i + j; & i > j \\ \frac{i}{j}; & i = j \\ i - j; & i < j \end{cases}$ (b) $a_{ij} = \begin{cases} 2^i; & i > j \\ i.j; & i = j \\ 3^j; & i < j \end{cases}$

(c) $a_{ij} = \begin{cases} i^2 + j^2; & i \neq j \\ 0; & i = j \end{cases}$ (d) $a_{ij} = \frac{|2i - 3j|}{5}$

(e) $a_{ij} = \left[\frac{i}{j}\right]$, where [.] represents Greatest Integer Function.

43. If $A = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$, then prove that $A^2 = \begin{pmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & \cos 2\theta \end{pmatrix}$, where $i = \sqrt{-1}$

44. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, evaluate $A^3 - 4A^2 + A$.

45. If $f(x) = \begin{pmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then prove that $f(x).f(y) = f(x + y)$

46. If $f(x) = \frac{1}{\sqrt{1 - x^2}}\begin{pmatrix} 1 & -x \\ -x & 1 \end{pmatrix}$, Prove that $f(x).f(y) = f\left(\frac{x + y}{1 + xy}\right)$. Hence show that $f(x).f(-x) = 1$, where $|x| < 1$.

FIVE MARKS QUESTIONS

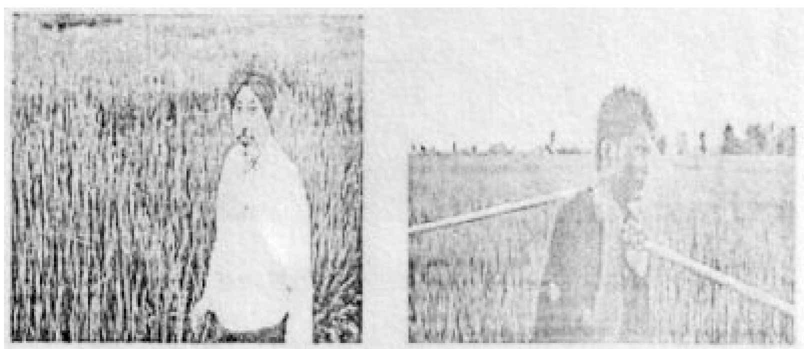
47. Find x, y and z if $A^T = A^{-1}$ and $A = \begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$. Also find how many triplets of (x, y, z) are possible. (NOTE: $A.A^{-1} = A^{-1}A = I$)

48. If A is a symmetric Matrix and B is skew-symmetric Matrix such that $A + B = \begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix}$ then show that $AB = \begin{pmatrix} 4 & -2 \\ -1 & -4 \end{pmatrix}$.

49. If $A = \begin{pmatrix} 4 & 1 \\ -9 & -2 \end{pmatrix}$ and $A^{50} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then show that $(a + b + c + d + 398) = 0$.

CASE STUDIES

50. (A) Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The Quantity of sale (in Kg) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.



$$A(\text{September sales}) = \begin{pmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{pmatrix}$$

(columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

$$B(\text{October sales}) = \begin{pmatrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{pmatrix}$$

(columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

If Ramakrishan sell the variety of rice (per kg)i.e. Basmati, Permal and Naura at Rs.30, Rs. 20 & Rs.10 respectively, While Gurcharan Singh sell the variety of rice (per kg) i.e. Basmati, Permal and Naura at Rs. 40, Rs. 30, & Rs.20 respectively.

Based on the above information answer the following:

(a) Find the Total selling Price received by Ramakrishan in the month of september.

(b) Find the Total Selling Price received by Gurcharan Singh in the month of september.

(c) Find the Total selling Price received by Ramakrishan in the month of september & october.

(B) A manufacture produces three stationery products Pencil, Eraser and Sharpener which he sells in two markets. Annual sales are indicated below

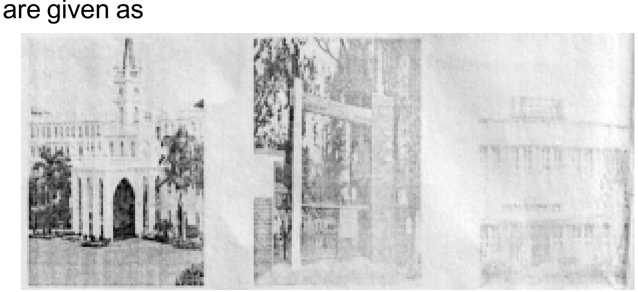
Market	Products (in numbers)		
	Pencil	Eraser	Sharpener
A	10,000	2000	18,000
B	600	20,000	8,00

If the unit Sale price of Pencil, Eraser and Sharpener are Rs. 2.50, Rs. 1.50 and Rs. 1.00 respectively, and unit cost of the above three commodities are Rs. 2.00, Rs. 1.00 and Rs. 0.50 respectively, then, Based on the above information answer the following:

(a) Find the total Revenue of both the markets.

(b) Find the total Profit for both the markets.

(C) Three schools ABC, PQR and MNO decided to organize a fair for collecting money for helping the flood victims. They sold handmade fans, mats and plates from recycled material at a cost of Rs. 25, Rs. 100 and Rs. 50 each respectively. The numbers of articles sold are given as



School/Article	ABC	PQR	MNO
Hand made fans	40	25	35
Marks	50	40	50
Plates	20	30	40

Based on the information given above, answer the following questions.

(a) What is the total amount of money (in Rs.) collected by all the three schools ABC, PQR & MNO?

(b) If the number of handmade fans and plates are interchanged for all the schools, then what is the total money collected by all schools?

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If A is a symmetric matrix then which of the following is not Symmetric matrix,

   (a) $A + A^T$ (b) $A.A^T$

   (c) $A - A^T$ (d) A^T

2. Suppose P, Q and R are different matrices of order 3 × 5, $a \times b$ and $c \times d$ respectively, then value of $ac + bd$ is, if matrix $P + Q - R$ is defined

   (a) 9 (b) 14

   (c) 24 (d) 34

3. If A and B are two square matrices of same order such that, $AB = A$ and $BA = B$, then $(A + B)(A - B) =$

   (a) O (b) A

   (c) $A^2 - B^2$ (d) B

4. If $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$, then $2x + y - z =$

   (a) 1 (b) 3

   (c) 5 (d) 7

5. If a matrix has 2022 elements, how many orders it can have?

   (a) 6 (b) 2

   (c) 4 (d) 8

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If matrix $A = [a_{ij}]_{2\times 2}$ where

   $a_{ij} = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$, then $A^{2021} =$

   (a) O (b) A

   (c) $-A$ (d) I

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then $A^4 =$

   (a) A (b) $3A$

   (c) $9A$ (d) $27A$

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 + pA + qI = 0$, then $pq =$

(a) 0 (b) 1

(c) –1 (d) 2

4. If $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$, then $a + b + c + d =$

(a) 0 (b) 4

(c) 6 (d) 10

5. If A is a square Matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(a) $2A + I$ (b) $A + 2I$

(c) I (d) $A + I$

ANSWER

One Mark Questions

1. (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2. (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. (b) + 1 4. (a) ±3

5. (b) 3 6. (d) 17

7. (a) 2023 × 2024 8. (d) 2024 × 2023

9. (a) 1 10. (a) 80

11. (b) 8 12. (d) 2

13. (d) 0 14. (b) 12

15. (b) x =2, y = 1 16. (a) $\begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$

17. (b) A 18. (c) 4

19. (a) I 20. (b) (14)

21. (d) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$ 22. (c) ±5

23. (b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 24. (d) A is false but R is true.

25. (a) Both A and R are true and R is the correct explanation of A

Two Marks Questions

31. $X = \begin{pmatrix} -3 & -14 \\ 4 & 17 \end{pmatrix}$

33. $(1) \to R \quad (2) \to S \quad (3) \to P \quad (4) \to Q$

35. (a) $\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 2 & 5 & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & -5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{5}{2} \\ 0 & 0 & \frac{11}{2} \\ \frac{-5}{2} & \frac{-11}{2} & 0 \end{pmatrix}$

35. (b) $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 7 \\ 5 & 7 & -5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

35. (c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$

37. $x = -2$, $y = 0$

38. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

39. $\begin{pmatrix} 29 & 24 \\ 6 & 5 \end{pmatrix}$

40. $\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

Three Marks Questions

42. (a) $\begin{pmatrix} 1 & -1 & -2 \\ 3 & 1 & -1 \\ 4 & 5 & 1 \end{pmatrix}$

42. (b) $\begin{pmatrix} 1 & 9 & 27 \\ 4 & 4 & 27 \\ 8 & 8 & 9 \end{pmatrix}$

42. (c) $\begin{pmatrix} 0 & 5 & 10 \\ 5 & 0 & 13 \\ 10 & 13 & 0 \end{pmatrix}$

40. (d) $\begin{pmatrix} \frac{1}{5} & \frac{4}{5} & \frac{7}{5} \\ \frac{1}{5} & \frac{2}{5} & 1 \\ \frac{3}{5} & 0 & \frac{3}{5} \end{pmatrix}$

42. (e) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$

44. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

47. $x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}};$

CASE STUDIES QUESTION

50. Case Study: A

(a) Rs. 1,00,000 (b) Rs. 3,10,000 (c) Rs. 5,10,000

50. Case Study: B

(b) Rs. 46,000 (For Market A)   (b) Rs. 15,000 (For Market A)

Rs. 43,000 (Ror Market B)   Rs. 17,000 (For Market A)

50. Case Study C: 50. (iv) Option (d)   50. (v) Option (c)

(a) Rs. 21,000

Rs. 21,250

SELF ASSESSMENT-1

1. (c)   2. (d)   3. (a)   4. (c)   5. (d)

SELF ASSESSMENT-2

1. (b)   2. (d)   3. (a)   4. (d)   5. (c)

CHAPTER-4

DETERMINANTS

One of the important aplications of inverse of a non-singular square matrix is in cryptography.

Cryptography is an art of communication between two people by keeping the information not known to others. It is based upon two factors, namely encryption and decryption.

Encryption means the process of transformation of an information (plain form) into an unreadable form (coded form). On the other hand, *Decryption* means the transformation of the coded message back into original form. Encryption and decryption require a secret technique which is known only to the sender and the receiver. This secret is called a *key*. One way of generating a key is by using a non-singular matrix to encrypt a message by the sender. The receiver decodes (decrypts) the message to retrieve the original mesage by using the inverse of the matrix. The matrix used for encryption is called encryption matrix (encoding matrix) and that used for decoding is called decryption matrix (decoding matrix).

TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM (2025-26)
• Determinant of a square matrix (up to 3 × 3 matrice), minors, co-factors and applications of determinants in finding the area of a triangle.
• Adjoint and inverse of a square matrix.
• Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

A determinant of order 2 is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where a, b, c, d are complex numbers (As Complex Number Include Real Number). It denotes the complex number $ad - bc$.

Even though the value of determinants Represented by Modulus symbol but the value of a determinant may be positive, negative or zero.

In other words,

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (Product of diagonal elements – Product of non-diagonal elements)

- Determinant of order 1 is the number itself.
- We can expand the determinants along any Row or Column, but for easier calculations we shall expand the determinant along that row or column which contains maximum number of zeroes.

MINORS AND COFATORS

Minor of an Element

If we take an element of the determinant and delete/remove the row and column containing that element, the determinant of the elements left is called the minor of that element. It is denoted by M_{ij}. For example,

Let us consider a Determinant $|A|$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} e & f \\ q & r \end{vmatrix} \text{ (Minor of } a_{11} = M_{11})$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{12} = \begin{vmatrix} d & f \\ p & r \end{vmatrix} \text{ (Minor of } a_{12} = M_{12})$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{13} = \begin{vmatrix} d & e \\ p & q \end{vmatrix} \text{ (Minor of } a_{13} = M_{13})$$

Hence a determinant of order two will have "4 minors" and a determinant of order three will have "9 minors".

Minor of an Element:

Cofactor of the element a_{ij} is $c_{ij} = (-1)^{i+j} M_{ij}$; where i and j denotes the row and column in which the particular element lies. (Means Magnitude of Minor and Cofactor of aij are equal).

- **Property:** If we multiply the elements of any row/column with their respective Cofactors of the same row/column, then we get the value of the determinant.

  For example,

  $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

  $|A| = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

- **Property:** If we multiply the elements of any row/column with their respective Cofactors of the other row/column, then we get zero as a result.

  For example,

  $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

Note that the value of a determinant of order three in terms of 'Minor' and 'Cofactor' can be written as:

$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ OR $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Clearly, we see that, if we apply the appropriate sign to the minor of an element, we have its Cofactor. The signs form a check-board pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

PROPERTIES OF DETERMINANTS

- The value of a determinant remains unaltered, if the row and solumns are inter changed.

$$|A| = |A^T|$$

$$\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

- If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only . e.g.

$$\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = -\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = \begin{vmatrix} b & y & q \\ a & x & p \\ c & z & r \end{vmatrix}$$

- If all the elements of a row (or column) are zero, then the value of determinant is zero.

$$\begin{vmatrix} a & 0 & x \\ b & 0 & y \\ c & 0 & z \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ x & y & z \end{vmatrix} = 0$$

- If the all elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.

$$\begin{vmatrix} a & ka & x \\ b & kb & y \\ c & kc & z \end{vmatrix} = \begin{vmatrix} mp & mq & mr \\ p & q & r \\ x & y & z \end{vmatrix} = 0$$

- If all the elements of a determinant above or below the main diagonal consist of zeros (Triangular Matrix), then the determinant is equal to the product of diagonal elements.

$$\begin{vmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{vmatrix} = \begin{vmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

- If all the elements of one row/column of a determinant are multiplied by "k" (A scalar), the value of the new determinant is k times the original determinant.

$$\begin{vmatrix} ka & p & x \\ kb & q & y \\ kc & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\begin{vmatrix} ka & kp & x \\ kb & kq & y \\ kc & kr & z \end{vmatrix} = k^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\begin{vmatrix} ka & kp & kx \\ kb & kq & ky \\ kc & kr & kz \end{vmatrix} = k^3 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$|kA| = k^n |A|$, where n is the order of determinant.

AREA OF A TRIANGLE

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ (sq. units)}$$

ADJOINT OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$ be a square matrix and C_{ij} be cofactor of a_{ij} in $|A|$.

Then, $(\text{adj } A) = [C_{ij}] \Rightarrow \text{adj } A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

- $A.(\text{adj } A) = (\text{adj } A).A = |A|\, I$
- $(\text{adj } AB) = (\text{adj } B).(\text{adj } A)$
- $|\text{adj } A| = |A|^{n-1}$, where n is the order of a Matrix A

SINGULAR MATRIX

A Matrix A is singular if $|A| = 0$ and it is non-singular if $|A| \neq 0$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5 \neq 0. \text{ So } A \text{ is Non-singular Matrix.}$$

$$|A| = \begin{vmatrix} 2 & 8 \\ 1 & 4 \end{vmatrix} = 8 - 8 = 0. \text{ So } A \text{ is singular Matrix.}$$

INVERSE OF A MATRIX

A square matrix A is said to be invertible if there exists a square matrix B of the same order such that $AB = BA = I$ then we write $A^{-1} = B$, (A^{-1} exists only if $|A| \neq 0$)

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

- $(AB)^{-1} = B^{-1}.A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $AA^{-1} = A^{-1}A = I$
- $|A^{-1}| = \dfrac{1}{|A|}$
- $|A.\text{adj } A| = |A|^n$ (Where n is the order of Matrix A)

Illustration:

For what value of k, the matrix $A = \begin{pmatrix} 2 & 10 \\ 5k-2 & 15 \end{pmatrix}$ is singular matrix.

Solution: As, Matrix is singular, so its determinant will be zero.

$|A| = 2(15) - 10(5k - 2) = 30 - 50k + 20$

$|A| = 50 - 50k = 0$

$\Rightarrow 50k = 50$

$\therefore \boxed{k = 1}$

Illustration:

Without expanding the determinants prove that $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Solution: Let $A = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

We observe here $a_{ij} = -a_{ji}$ (A is skew-symmetric matrix)

$\Rightarrow A^T = -A$

$\Rightarrow |A^T| = |-A|$

$\Rightarrow |A| = (-1)^3 |A|$

Property USED: $|A^T| = |A|$, $|kA| = k^n|A|$

Where n is the order of the determinant

$\Rightarrow |A| = -|A|$

$\Rightarrow 2|A| = 0$

$$\Rightarrow |A| = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Illustration:

If A is an invertible matrix of order 2 and $|A| = 4$, then write the value of $|A^{-1}|$.

Solution: As we know that,

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}$$

$$\Rightarrow \boxed{|A^{-1}| = \frac{1}{4}}$$

Illustration:

Find the inverse of the matrix $\begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{pmatrix}$ and hence solve the system of equations:

$3x + 4y + 5z = 18$

$5x - 2y + 7z = 20$

$2x - y + 8z = 13$

Solution: Let, $A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{pmatrix}$

Cofactors are,

$C_{11} = \begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} = -7 + 16 = 9$ $\qquad C_{21} = -\begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} = -38$ $\qquad C_{31} = \begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} = 37$

$C_{12} = -\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} = -(14 - 40) = 26$ $\qquad C_{22} = \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -4$ $\qquad C_{32} = -\begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} = -14$

$C_{13} = \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -4 + 5 = 1$ $\qquad C_{23} = -\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = 26$ $\qquad C_{33} = \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11$

$$\text{Adj } A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{pmatrix}$$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 3(9) + 4(26) + 5(1) = 27 + 104 + 5 = 136$

$$\text{So, } A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{136}\begin{pmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{pmatrix}$$

Given system of equation can be written as

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 20 \\ 13 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 3 & 4 & 5 \\ 5 & -2 & 7 \\ 2 & -1 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 13 \\ 20 \end{pmatrix}$$

$\Rightarrow A.X = B \Rightarrow A^{-1} AX = A^{-1}.B$

$IX = A^{-1}.B \Rightarrow X = A^{-1}.B$

$$X = \frac{1}{136}\begin{pmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{pmatrix}\begin{pmatrix} 18 \\ 13 \\ 20 \end{pmatrix} = \frac{1}{136}\begin{pmatrix} 9\times 18 - 38\times 13 + 37\times 20 \\ 26\times 18 - 4\times 13 - 14\times 20 \\ 1\times 18 + 26\times 13 - 11\times 20 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{136}\begin{pmatrix} 408 \\ 136 \\ 136 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ So, $x = 3$, $y = 1$, $z = 1$

ONE MARK QUESTIONS

1. If f(x)=$\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$, then determinant of $\left(f\left(\frac{\pi}{6}\right).f\left(\frac{\pi}{3}\right)\right) =$

(a) 0 (b) 1 (c) –1 (d) $\frac{\pi}{2}$

2. If for a square matrix $A, A^2 - A + I = 0$, then A^{-1} equal

(a) A (b) I + A (c) A – I (d) I – A

3. If $\begin{vmatrix} x & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix}$ = 0, then value of x is

(a) 0 (b) 1 (c) 4 (d) 2

4. The value of $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ is

(a) xyz (b) (x + y + z) (c) 2(x + y + z) (d) 0

5. If $A = \begin{pmatrix} 2 & 2023 & 2024 \\ 0 & 1 & 2022 \\ 0 & 0 & 5 \end{pmatrix}$, then A. (adj A) equals

(a) 2I (b) I (c) 5I (d) 10 I

6. If $A = \begin{pmatrix} 3 & 1 \\ 19 & 7 \end{pmatrix}$, then A.(adj A) equals

(a) $\begin{pmatrix} 3 & 1 \\ 19 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & -1 \\ -19 & 3 \end{pmatrix}$

7. If the area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units, then |k| =

(a) 0 (b) 6 (c) 3 (d) 9

8. If the area of a triangle with vertices (2, –6), (5, 4) and (k, 4) is 35sq. units, then the sum of all possible values of k is

(a) 2 (b) 10 (c) 12 (d) 14

9. If $A=\begin{pmatrix}2023 & 1\\ 2024 & 1\end{pmatrix}$, then A^{-1} =

(a) $\begin{pmatrix}-2023 & 1\\ 2024 & -1\end{pmatrix}$ (b) $\begin{pmatrix}-1 & 1\\ 2024 & -2023\end{pmatrix}$

(c) $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ (d) $\begin{pmatrix}-1 & -1\\ -2024 & -2023\end{pmatrix}$

10. If $A=\begin{pmatrix}k & 16\\ -9 & -k\end{pmatrix}$ is singular matrix, then sum of all possible values of k is

(a) 0 (b) 12 (c) 10 (d) 24

11. If $A=\begin{pmatrix}k & 12\\ 3 & 6\end{pmatrix}$ is non-invertible matrix, then value of k is

(a) 0 (b) 3 (c) 6 (d) 12

12. If $A.(adjA)=\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{pmatrix}$, then $|A|+|adjA|$ =

(a) 5 (b) 10 (c) 25 (d) 30

13. If $A.(adjA)=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$, then $\dfrac{|A|+|A^T|}{|A^{-1}|}$ =

(a) 2 (b) 8 (c) 4 (d) 16

ASSERTION-REASON BASED QUSTIONS (Q. 14 & Q.15)

In the following questions, a statement of assertion (A) is following by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A are R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explnantion of A.

(c) A is true but R is false

(d) A is false but R is true

14. Assertion: For Matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, value of $4C_{31} + 5C_{32} + 6C_{33}$ is 0.

Reasoning : The sum of the products of elements of any row of a matrix A with the co-factors of elements of other row is always equal to Zero.

15. Assertion : If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then determinant of matrix A is zero.

Reasoning : The determinant of a skew-symmetric matrix of order 3 × 3 is always zero.

TWO MARKS QUESTIONS

16. Without expanding the determinants prove that $\begin{vmatrix} 0 & 2023 & -2021 \\ -2023 & 0 & -2022 \\ 2021 & 2022 & 0 \end{vmatrix} = 0$

17. Let A be a 3 × 3 matrix such that $|A| = -2$, then find the value of $|-2A^{-1}| + 2|A|$.

18. If $A = \begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}, B = \begin{pmatrix} yr - zq & cq - br & bz - cy \\ zp - xr & ar - cp & cx - az \\ xq - yp & bp - aq & ay - bx \end{pmatrix}$. Find $|B|$ if $|A| = 4$

19. If $A = \begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}, B = \begin{pmatrix} yr - zq & cq - br & bz - cy \\ zp - xr & ar - cp & cx - az \\ xq - yp & bp - aq & ay - bx \end{pmatrix}$. Find $|A|$ if $|B| = 25$

20. Find the Adjoint of Matrix A,

$$A = \begin{pmatrix} 2\cos\frac{\pi}{3} & -2\sin\frac{\pi}{3} \\ 2\sin\frac{\pi}{3} & -2\cos\frac{\pi}{3} \end{pmatrix}$$

THREE MARKS QUESTIONS

21. If A is a square matrix of order 3, such that $|\text{Adj } A| = 25$, then find the value of

(a) $|A|$ (b) $|-2A^T|$ (c) $|4A^{-1}|$

(d) $|5A|$ (e) $A.\text{Adj } A$ (f) $|A.\text{Adj } A|$

(g) $|A^3|$

22. If A is a square matrix of order 3, such that $|A| = 5$, then find the value of

(a) $|3A|$ (b) $|-2A^T|$ (c) $|4A^{-1}|$

(d) $|\text{Adj } A|$ (e) $A.\text{Adj } A$ (f) $|A.\text{Adj } A|$

(f) $|A^3|$

23. If $A = \begin{pmatrix} 1 & 2020 & 2021 \\ 0 & 1 & 2022 \\ 0 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 0 \\ 2021 & 1 & 0 \\ 2020 & 2022 & 1 \end{pmatrix}$ then find the value of

(a) $|AB|$ (b) $|(AB)^{-1}|$ (c) $|A^2.B^3|$

(d) $|3(AB)^T|$ (e) $|\text{Adj }(AB)|$

24. Find matrix 'X' such that $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} X \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$

25. Find matrix 'X' such that

(a) $X \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} X \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

FOUR/FIVE MARKS QUESTIONS

26. (a) A school wants to award its students for regularity and hardwork with a total cash award of ₹ 6,000. If three times the award money for hardwork added to that given for regularity amounts of ₹ 11,000 represent the above situation algebraically and find the award money for each value, using matrix method.

(b) A shopkeeper has 3 varieties of pen A, B and C. Rohan purchased 1 pen of each variety for total of ₹ 21. Ayush purchased 4 pens of A variety, 3 pens of B variety and 2 pen of C variety for ₹ 60. While Kamal purchased 6 pens of A variety, 2 pens of B variety and 3 pen of C variety for ₹ 70. Find cost of each variety of pen by Matrix Method.

27. Find A^{-1}, where $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}$. Hence use the result to solve the following system of linear equations:

$$x + 2y - 3z = -4$$
$$2x + 3y + 2z = 2$$
$$3x - 3y - 4z = 11$$

28. Find A^{-1}, where $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}$. Hence, solve the system of linear equations:

$$x + 2y + 3z = 8$$
$$2x + 3y - 3z = -3$$
$$-3x + 2y - 4z = -6$$

29. If $A = \begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix}$ find AB. Hence using the product solve the system of eq.

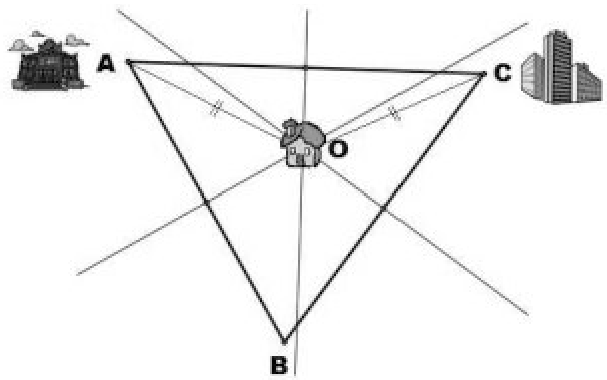
$$x - y + z = 4$$
$$x - 2y - 2z = 9$$
$$2x + y + 3z = 1$$

30. Find the product of matrices AB, where $A = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}$ and use the result to solve following system of equations:

$$x - 2y - 3z = 1$$
$$-2x + 4y + 5z = -1$$
$$-3x + 7y + 9z = -4$$

CASE STUDY BASED QUESTIONS

A. A family wanted to buy a home, but they wanted it to be close both to both the children's school and the parents' workplace. By looking at a map, they cold find a point that is equidistant from both the workplace and the school by finding the *circumcenter* of the triangular region.



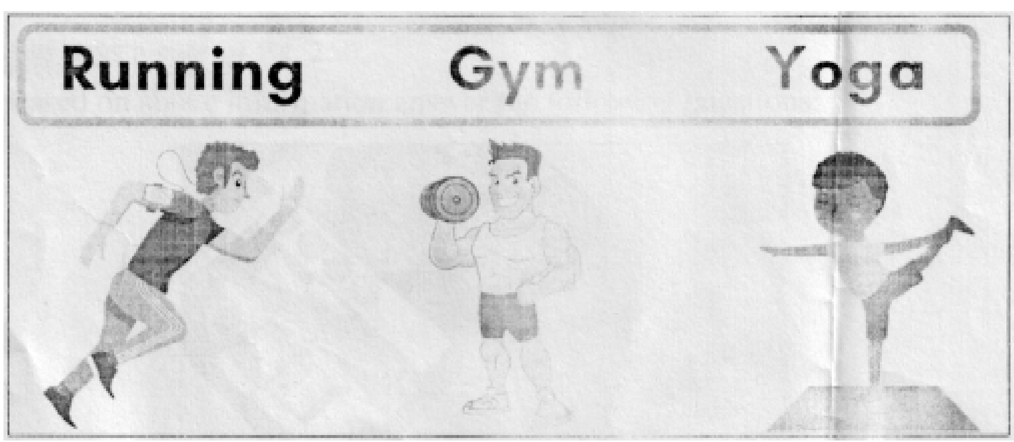
If the coordinates are $A(12, 5)$, $B(20, 5)$ and $C(16, 7)$, on the basis of this answer the following: (Figure is for reference only, Not as per scale)

(a) Using the concept of Determinants. Find the equation of AC.

(b) If any point $P(2, k)$ is collinear with point $A(12, 5)$ and $O(16, 2)$, then find the value of $(2k – 15)$.

(c) If any point P(2, k) is collinear with point A (12, 5) and 0 (16, 2), then find the value of $(2k – 15)$.

B. For keeping Fit, X people believes in morning walk, Y people believes in yoga and Z people join Gym. Total no of people are 70. Further 20%, 30% and 40% people are suffering from any disease who believe in morning walk, yoga and GYM respectively. Total no. of such people is 21. If morning walk cost ₹ 0 Yoga cost ₹ 500/month and GYM cost ₹ 400/ month and total expenditure is ₹ 23000.



On the basis of above information, answer the following:

(a) If matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 4 \end{pmatrix}$, then find A^{-1}.

(b) On solving the given situational problem using matrix method, find the total number of person who prefer GYM.

C. An amount of ₹ 600 crores is spent by the government in three schemes. Scheme *A* is for saving girl child from the cruel parents who don't want girl child and get the abortion before her birth.

Scheme *B* is for saving of newlywed girls from death due to dowry. Scheme *C* is planning for good health for senior citizen. Now twice the amount spent on Scheme *C* together with amount spent on Scheme *A* is ₹ 700 crores. And three times the amount spent on Scheme *A* together with amount spent on Scheme *B* and Scheme *C* is ₹ 1200 crores.

If we assume government invest (In crores) ₹ X, ₹ Y and ₹ Z in scheme *A*, *B* and *C* respectively. Solve the above problem using Matrices and answer the following:

C. Gautam buys 5 pens, 3 pens, 3 bags & 1 instrumental box and pays a sum of Rs. 160. From the same shop, Vikram buys 2 pens, 1 bag & 3 instrumental boxes and pays a sum of Rs. 190. Also Ankur buys 1 pen, 2 bags & 4 instrumental boxes and pays a sum of Rs. 250.

Based on above informatin answer the following questions:

(a) Convert the given situation into a matrix equation of the form AX = B.

(b) Find |A|.

(c) Find A^{-1}.

OR

Determine $P = A^2 - 5A$

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $A = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 8 \\ 0 & 0 & 5 \end{bmatrix}$, then $|A|$ =

   (a) 2 (b) 5

   (c) 8 (d) 10

2. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$, then $|A^T|$ =

   (a) 2 (b) 5

   (c) 8 (d) 10

3. If $A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, then $|A^{-1}|$ =

   (a) 0 (b) 1

   (c) $\cos x . \sin x$ (d) –1

4. If $A = \begin{bmatrix} 6x & 8 \\ 3 & 2 \end{bmatrix}$ is singular matrix, then the value of x is

   (a) 2 (b) 3

   (c) 5 (d) 7

5. The area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units. The value of k will be

   (a) 6 (b) 9

   (c) 3 (d) 0

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE

1. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be

   (a) 0 (b) 1

   (c) 12 (d) 144

2. If the points (3, –2), (x, 2), (8, 8) are collinear, then x =

   (a) 2 (b) 5

   (c) 4 (d) 3

3. $\begin{vmatrix} \cos 15^\circ & \sin 75^\circ \\ \sin 15^\circ & \cos 75^\circ \end{vmatrix} =$

(a) 0 (b) 1

(c) –1 (d) 2

4. The minor of 6 in the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ is

(a) 9 (b) –6

(c) 6 (d) 10

5. The cofactor of 4 in the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ is

(a) 9 (b) –6

(c) 6 (d) 10

ANSWER

One Mark Questions

1. (b) 1
2. (d) I – A
3. (c) 4
4. (d) 0
5. (d) 10I
6. (c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
7. (c) 3
8. (b) 10
9. (b) $\begin{pmatrix} -1 & 1 \\ 2024 & -2023 \end{pmatrix}$
10. (a) 0
11. (c) 6
12. (d) 30
13. (b) 8
14. (a) Both A and R are true and R is the correct explanation of A.
    (b) Both A and R are true and R is not the correct explanation of A.

Two Marks Questions

17. 0
18. 16
19. ±5
20. $\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

Three Marks Questions

21. (a) ±5 (b) ±40 (c) $\frac{\pm 64}{5}$ (d) ±625 (e) ±5I
    (f) ±125 (g) ±125
22. (a) 135 (b) –40 (c) $\frac{64}{5}$ (d) 25 (e) 5I
    (f) 125 (e) 125
23. (a) 6 (b) $\frac{1}{6}$ (c) 72 (d) 162 (e) 36
24. $X = \frac{1}{9}\begin{pmatrix} 2 & 31 \\ -1 & -11 \end{pmatrix}$
25. (a) $X = \begin{pmatrix} 16 & -25 \\ 1 & -1 \end{pmatrix}$ (b) $X = \begin{pmatrix} 11 & -7 \\ -5 & 4 \end{pmatrix}$ (c) $X = \frac{1}{9}\begin{pmatrix} 5 & -17 \\ -3 & 12 \end{pmatrix}$

Five Marks Questions

26. (a) Award money given for
    Honesty = ₹ 500
    Regularity = ₹ 2000 and

Hard work = ₹ 3500

(b) Cost of pen of

Variety A = ₹ 5

Variety B = ₹ 8 and

Variety C = ₹ 8

27. $x = 3, y = -2, z = 1$ 28. $x = 0, y = 1, z = 2$ 29. $x = 3, y = -2, z = -1$

30. $x = -4, y = -1, z = -1$

CASE STUDIES QUESTIONS

A. (a) x – 2y = 2 (b) 10 sq. units (c) 10

B. (a) $A^{-1} = \frac{-1}{6}\begin{pmatrix} -8 & 1 & 1 \\ -8 & 4 & -2 \\ 10 & -5 & 1 \end{pmatrix}$ (b) 20

C. (a) $\begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 160 \\ 190 \\ 250 \end{pmatrix}$

A X = B

(b) –22

(c) $\frac{1}{-22}\begin{pmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{pmatrix}$

OR PART

$\begin{pmatrix} 7 & 5 & 13 \\ 5 & 8 & 2 \\ 8 & 3 & 3 \end{pmatrix}$

SELF ASSESSMENT-1

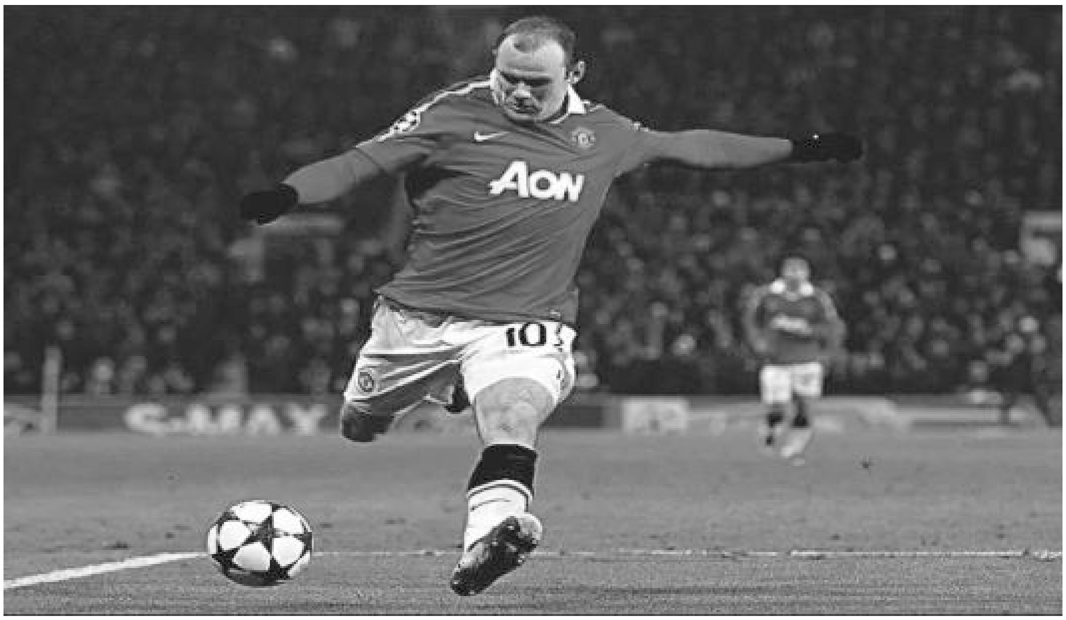
1. (d) 2. (d) 3. (b) 4. (a) 5. (c)

SELF ASSESSMENT-2

1. (d) 2. (b) 3. (a) 4. (b) 5. (c)

CHAPTER 5

CONTINUITY AND DIFFERENTIABILITY



Many real life events, such as trajectory traced by Football where you see player hit the soccer ball, angle and the distance covered animation on the screen is shown to the viewers using technology can be described with the help of mathematical functions. The knowledge of Continuity and differentiation is popularly used in finding speed, directions and other parameters from a given function.

CONTINUITY AND DIFFERENTIABILITY

Topics to be covered as per C.B.S.E. revised syllabus (2025-26)

- Continuity and differetiability
- Chain rule
- Derivative of inverse trigonometric functions, like $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$
- Derivative of implicit functions.
- Concept of exponential and logarithmic function
- Derivatives of logarithmic and exponential functions.
- Logarithmic differentiation, derivative of functions expressed in parametric forms.
- Second order derivatives.

POINTS TO REMEMBER

- A function $f(x)$ is said to be continuous at $x = c$ iff $\lim\limits_{x \to c} f(x) = f(c)$

  i.e., $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = f(c)$

- $f(x)$ is continuous in (a, b) iff it is continuous at $x = c \ \forall c \in (a, b)$.

- $f(x)$ is continuous in $[a, b]$ iff

  (i) f(x) is continuous in (a, b)

  (ii) $\lim\limits_{x \to a^+} f(x) = f(a)$

  (iii) $\lim\limits_{x \to b^-} f(x) = f(b)$

- Modulus functions is Continuous on R
- Trigonometric functions are continuous in their respective domains.
- Exponential function is continuous on R
- Every polynomial function is continuous on R.
- Greatest integer function is continuous on all non-integral real numbers
- If $f(x)$ and $g(x)$ are two continuous functions at $x = a$ and if $c \in R$ then

  (i) $f(x) \pm g(x)$ are also continuous functions at $x = a$.

  (ii) $g(x) . f(x)$, $f(x) + c$, $cf(x)$, $|f(x)|$ are also continuous at $x = a$.

  (iii) $\frac{f(X)}{g(X)}$ is continuous at $x = a$, provided $g(a) \neq 0$.

- A function $f(x)$ is derivable or differentiable at $x = c$ in its domain iff

  $\lim\limits_{x \to c^-} \frac{f(X) - f(c)}{X - c} = \lim\limits_{x \to c^+} \frac{f(X) - f(c)}{X - c}$, and is finite

  The value of above limit is denoted by $f'(c)$ and is called the derivative of $f(x)$ at $x = c$.

  $\frac{d}{dx}(u \pm v) = \frac{du}{dx} + \frac{dv}{dx}$

- $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v.\frac{du}{dx}$ (Product Rule)

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$ (Quotient Rule)

- If y= f(u) and u=g(t) then $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = f'(u)g'(t)$ (Chain Rule)

- If y =f(u), x=g(u) then,

  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{f'(u)}{g'(u)}$

Illustration:

Discuss the continuity of the function *f*(*x*) given by

$f(x) = \begin{cases} 4 - x, & x < 4 \\ 4 + x, & x \geq 4 \end{cases}$ at $x = 4$

Solution: We have *f*(*x*) = $\begin{cases} 4 - x, & x < 4 \\ 4 + x, & x \geq 4 \end{cases}$

$\text{LHL} = \lim_{x \to 2^{/}} f(x) = \lim_{x \to 4^-} (4 - x) = \lim_{h \to 0^-} 4 - (4 - h) = 0$

$\text{RHL} = \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (4 + x) = \lim_{h \to 0^+} 4 + (h + 4) = 8 + 0 = 8$

Here LHL ≠ RHL

Hence *f*(*x*) is not continuous at *x* = 4

Illustration:

Show that the function *f*(*x*) given by

$f(x) = \begin{cases} \frac{\tan x}{x} + \cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$ is continuous at $x = 0$

Solution: We have *f*(*x*) = $\begin{cases} \frac{\tan x}{x} + \cos x, & x \neq 4 \\ 2, & x = 4 \end{cases}$

Now *f*(0) = 2 ... (i)

$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\tan x}{x} + \cos x \right) = \lim_{h \to 0^-} \frac{\tan(0 - h)}{(o - h)} + \cos(0 - h) = \lim_{h \to 0} \left[\frac{-\tan h}{-h} + \cos h \right]$

$= \lim_{h \to 0} \frac{\tan h}{h} + \lim_{h \to 0} \cos h = 1 + \cos(0) = 1 + 1 = 2 \quad ...(ii)$

$$\text{RHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \left(\frac{\tan x}{x} + \cos x \right) = \lim_{h \to 0^+} \frac{\tan(0+h)}{(0+h)} + \cos(0+h)$$

$$= \lim_{h \to 0} \frac{\tan h}{h} + \lim_{h \to 0} \cos h = 1 + \cos(0) = 1 + 1 = 2 \qquad ...(iii)$$

$$\text{LHL} = \text{RHL} = f(0)$$

Hence f(x) is continuous at $x = 0$

ONE MARK QUESTIONS

Continuity and Differentiability

This section comprises Multiple Choice Questions (MCQ) of one mark each

1. The value of k for which the function f given by

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & if\ x \neq \dfrac{\pi}{2} \\ 5, & if\ x = \dfrac{\pi}{2} \end{cases}$$ is continuous at $x = \dfrac{\pi}{2}$ is :

(a) 6 (b) 5

(c) $\dfrac{5}{2}$ (d) 10

2. The value of k for which

$$f(x) = \begin{cases} 3x + 5, x \geq 2 \\ k\ x^2, x < 2 \end{cases}$$ is a continuous function is :

(a) $\dfrac{-11}{4}$ (b) $\dfrac{4}{11}$

(c) 11 (d) $\dfrac{11}{4}$

3. For what value of k, may the function $f(x) = \begin{cases} k(3x^2 - 5x), x \leq 0 \\ \cos x, x > 0 \end{cases}$ becomes continuous ?

(a) 0 (b) 1

(c) $\dfrac{-1}{2}$ (d) No value

4. If $f(x)=\begin{cases}\dfrac{\sin \pi x}{5x}, x\neq 0\\ k, x=0\end{cases}$ is continuous at x = 0, then k is equal to :

(a) $\dfrac{5}{\pi}$ (b) $\dfrac{\pi}{5}$

(c) 1 (d) 0

5. if $f(x)=\begin{cases}\dfrac{\sqrt{x^2+5-3}}{x+2}, x\neq -2\\ k \quad , x=-2\end{cases}$ is continuous at x = –2 , then the value of k is equal to :

(a) $\dfrac{-2}{3}$ (b) 0

(c) $\dfrac{2}{3}$ (d) none of these

6. If $f(x)\begin{cases}\dfrac{\tan\left(\dfrac{\pi}{4}-x\right)}{\cot 2x}, x\neq \dfrac{\pi}{4}\\ k \quad , x=\dfrac{\pi}{4}\end{cases}$ is continuous at x = $\dfrac{\pi}{4}$, then the value of k is :

(a) 1 (b) 2

(c) $\dfrac{1}{2}$ (d) none of these

7. The number of points of discontinuity of the rational function $f(x)=\dfrac{x^2-3x+2}{4x-x^3}$ is :

(a) 1 (b) 2

(c) 3 (d) none of these

8. The function f(x) =[x], where [x] denotes the greatest integer function, is continuous at x =

(a) –2 (b) 1

(c) 4 (d) 1.5

9. The function f(x) = |x| at x = 0 is :

(a) continuous but not differentiable

(b) diffcrcntiablc but not continuous

(c) continuous and differentiable

(d) neither continuous nor differentiable

10. The function f(x) = |x| + |x-1| is :

(a) differentiable at x = 0 but not at x = I

(b) differentiable at x = I but not at x = 0

(c) neither differentiable at x = 0 nor at x = 1

(d) differentiable at x = 0 as well as at x = 1

11. The set of numbers where the function f given by $f(x) = |2x - 1| \cos x$ is differentiable is:

(a) R (b) $R-\left(\frac{1}{2}\right)$

(c) $(0,\infty)$ (d) none of these

12. If $y = \log\left(\frac{1-x^2}{1+x^2}\right), |x| < 1$ then $\frac{dy}{dx} =$

(a) $\frac{4x^3}{1-x^4}$ (b) $\frac{-4x}{1-x^4}$

(c) $\frac{1}{4-x^4}$ (d) $\frac{-4x^3}{1-x^4}$

13. The derivative of $\sec(\tan^{-1}x)$ w.r.t. x is

(a) $\frac{x}{1+x^2}$ (b) $\frac{1}{\sqrt{1+x^2}}$

(c) $\frac{x}{\sqrt{1+x^2}}$ (d) $x\sqrt{1+x^2}$

14. If $y = \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \operatorname{cosec}^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)$ then $\frac{dy}{dx}$ is equal to :

(a) $\frac{\pi}{2}$ (b) 0

(c) 1 (d) none of these

15. Differential of $\log[\log(\log x^5)]$ w.r.t. x is :

(a) $\frac{5}{x\log(x^5)\log(\log x^5)}$ (b) $\frac{5}{x\log(\log x^5)}$

(c) $\frac{5x^4}{\log(x^5)\log(\log x^5)}$ (d) $\frac{5x^4}{\log(\log x^5)}$

16. If $y = \sin(m \sin^{-1} x)$ then which of the following equations is true?

(a) $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + m^2y = 0$

(b) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

(c) $(1+x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

(d) $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$

17. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to :

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{\cos x}{1-2y}$

(c) $\frac{\sin x}{1-2y}$ (d) $\frac{\sin x}{2y-1}$

Q no (18-22) are Assertion Reason Based questions carrying one mark each. These type of questions consists of two statements , one labelled Assertion (A) and other labelled Reason (R). Select the correct answer from the codes (a), (b),(c), and (d) as given below.

(a) Bolh Assertion (A) and Reason (R)arc true and Reason (R) is the correct explanation of the Assertion (A).

(b) Both Assertion (A) and Reason (R)arc true and Reason (R) is **not** the correct explanation of the Assertion (A).

(c) Assertion (A)is true and Reason(R) is false.

(d) Assertion (A)is false and Reason(R) is true

18. Let $f(x) = \frac{1}{1-x} - \frac{3}{1-x^3}, x \neq 1$

Statement -I: The value of f(1)so that f is continuous function is 1

Statement-II : $g(x) = \frac{x+2}{x^2+x+1}$ is continuous function

Answer (d) Assertion (A)is false and Reason(R) is true

19. Consider the function f(x) = |x – 2| + |x – 5|, $x \in R$

Statement - I : f'(4)=0

Statement -II : f is continuous on [2, 5] differentiable on (2, 5) and f(2) = f(5)

Solution (b) Both Assertion (A) and Reason (R)are true and Reason (R) is **not** the correct explanation of the Assertion (A).

20. **Statement -I :** $f(x)=\begin{cases} x^2 \sin\frac{1}{x}, x \neq 0 \\ 0, x=0 \end{cases}$

Statement-II : Both $h(x) = x^2$ and $g(x)=\begin{cases} \sin\frac{1}{x}, x \neq 0 \\ 0, x=0 \end{cases}$ continuous at $x = 0$

21. F(x) is defined as the product of two real functions $f_1(x) = x \ \ \forall\ x \in R$ and

$f_2(x)=\begin{cases} x \sin\frac{1}{x}, x \neq 0 \\ 0, x=0 \end{cases}$ as follows

$$F(x)=\begin{cases} f_1(x).f_2(x) \text{ if } x \neq 0 \\ 0, \text{ if } x=0 \end{cases}$$

Statement -I: F(x) is continuous on R

Statement-II : $f_1(x)$ and $f_2(x)$ are continuous on R

22. Let f(x) be a differentiable function such that $f(2)=4$ and $f'(2)=4$

Statement -I: $\lim_{x \to 2} \frac{xf(2)-2f(x)}{x-2} = -4$

Statement -II : $f'(a) = \lim_{x \to a} = \frac{f(x)-f(a)}{x-a}$

CASE BASED

23. A potter made a mud vessel , where the shape of pot is based on f(x) = |x-3| + |x-2|, where f(x) represents the height of the pot.

Based on the information given above answer the following questions

(a) When x > 4 what will be the height in terms of x?

(b) When the value of x lies between (2, 3) then find the value of f(x).

(c) If the potter is trying to make pot using the function f(x)=[x], will he get a pot or not? why?

Q24. Let x = f(t) and y = g(t) be the parametric forms with t as parameter, then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$ where $f'(t) \neq 0$

On the basis of the above information answer the following questions :

(a) What will be the derivative of f(tanx) w.r.t g(secx) at $x = \frac{\pi}{4}$ where $f'(1)$ and $g'(\sqrt{2}) = 4$?

(b) Find the derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t $\cos^{-1} x$.

(c) If y = $\frac{1}{4}u^4$ and $u = \frac{2}{3}x^3$ then find $\frac{dy}{dx}$

25. A function f(x) is said to be differentiable at x=c if

(i) Left hand derivative (L.H.D) = $f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$ exists finitely.

(ii) Right hand derivative (R.H.D) = $f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$ exists finitely.

(iii) R.H.D = L.H.D, i.e. if the function f(x) is differentiable at x = c, then f' (c) = $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

Based on the above information answer the following :

(a) If f(x) is differentiable at x = 3. then find the value of $\lim_{h \to 3} \frac{x^2 f(3) - 9f(x)}{x - 3}$

(b) Find $\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{h}$ if it exists.

TWO MARKS QUESTIONS

1. Differentiate sin (x^2) w. r. t. $e^{\sin x}$

2. $y = x^y$ then find $\frac{dy}{dx}$

3. If $y = x^x + x^3 + 3^x + 3^3$, find $\frac{dy}{dx}$

4. If $y = 2\sin^{-1}(\cos x) + 5\,\text{cosec}^{-1}(\sec x)$. Find $\frac{dy}{dx}$

5. If $y = e^{[\log(x+1) - \log x]}$ find $\frac{dy}{dx}$

6. Differentiate $\text{Sin}^{-1}[x\sqrt{x}]$ w. r. t. x.

7 Find the derivative of $|x^2+2|$ w.r.t. x

8. Find the domain of the continuity of $f(x) = \sin^{-1}x - [x]$

9. Find the derivative of $\cos(\sin x^2)$ w.r.t. x at $x = \sqrt{\frac{\rho}{2}}$

10. If $y=e^{3\log x + 2x}$, Prove that $\frac{dy}{dx} = x^2(2x+3)\,e^{2x}$.

11. Differentiate $\sin^2(\theta^2+1)$ w.r.t. θ^2

12. Find $\frac{dy}{dx}$ if $y = \sin^{/1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)$

13. If $x^2 + y^2 = 1$ verify that $\frac{dy}{dx}.\frac{dx}{dy} = 1$

14. Find $\frac{dy}{dx}$ when $y = 10^{x10^x}$

15. If $y = x^x$ find $\frac{d^2y}{dx^2}$

16. Find $\frac{dy}{dx}$ if $y = \cos^{-1}(\sin x)$

17. If f(x) = x + 7, and g(x) = x – 7, $x \in R$, them find $\frac{d}{dx}$ (fog) (x).

18. Differntiate log (7 logx) w.r.t x

19. If $y = f(x^2)$ and $f'(x) = \sin x^2$. Find $\frac{dy}{dx}$

20. Find $\frac{dy}{dx}$ if $y = \sqrt{\sin^{-1}\sqrt{x}}$

THREE MARKS QUESTIONS

1. Examine the continuity of the following functions at the indicated points.

(I) $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & , x = 0 \end{cases}$ at $x = 0$

(II) $f(x) = \begin{cases} x - [x], & x \neq 1 \\ 0 & , x = 1 \end{cases}$ at $x = 1$

(III) $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ at $x = 0$

(IV) $f(x) = \begin{cases} \dfrac{x - \cos(\sin^{-1}x)}{1 - \tan(\sin^{-1}x)} & x \neq \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & x = \frac{1}{\sqrt{2}} \end{cases}$ at $x = \frac{1}{\sqrt{2}}$

2. For what values of constant K, the following functions are continuous at the indicated points.

(i) $f(x) = \begin{cases} \dfrac{\sqrt{1+Kx} - \sqrt{1-Kx}}{x} & x < 0 \\ \dfrac{2x+1}{x-1} & x > 0 \end{cases}$ at $x = 0$

(ii) $f(x) = \begin{cases} \dfrac{e^x - 1}{\log(1+2x)} & x \neq 0 \\ K & x = 0 \end{cases}$ at $x = 0$

(iii) $f(x) = \begin{cases} \dfrac{1 - \cos 4x}{x^2} & x < 0 \\ K & x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & x > 0 \end{cases}$ at $x = 0$

3. For what values a and b

$$f(x) = \begin{cases} \dfrac{x+2}{|x+2|} + a & if\ x < -2 \\ a+b & if\ x = -2 \\ \dfrac{x+2}{|x+2|} + 2b & if x > -2 \end{cases}$$

Is continuous at $x = -2$

4. Find the values of a, b and c for which the function

$$f(x) = \begin{cases} \dfrac{\sin[(a+1)x] + sinx}{x} & x < 0 \\ C & x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & x > 0 \end{cases}$$

Is continuous at $x = 0$

5. $f(x) = \begin{cases} [x] + [-x] & x \neq 0 \\ \lambda & x = 0 \end{cases}$

Find the value of λ, *f* is continuous at $= 0$?

6. Let $f(x) = \begin{cases} \frac{1-sin^3 x}{3cos^2 x} & ; & x < \frac{\pi}{2} \\ a & ; & x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2} & ; & x > \frac{\pi}{2} \end{cases}$

If $f(x)$ is continuous at $x = \frac{\pi}{2}$, find *a* and *b.*

7. If $f(x) = \begin{cases} x^3 + 3x + a & x \leq 1 \\ bx + 2 & x > 1 \end{cases}$

Is everywhere differentiable, find the value of *a* and *b*.

8. Find the relationship between a and b so that the function defined by

$$f(x) = \begin{Bmatrix} ax+1 \ , & x \leq 3 \\ bx+3 \ , & x > 3 \end{Bmatrix} is\ continous\ at\ x = 3.$$

9. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$ where $x \neq 0$.

10. If $y = x^{x^x}$, then find $\frac{dy}{dx}$.

11. Differentiate $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$ w.r.t. x.

12. If $(x+y)^{m+n} = x^m . y^n$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

13. If $(x-y). e^{\frac{x}{x-y}} = a$, prove that $y\left(\frac{dy}{dx}\right) + x = 2y$

14. If $x = \tan\left(\frac{1}{a}\log y\right)$ then show that

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$$

15. If $y = x\log\left(\frac{x}{a+bx}\right)$ prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

16. Differentiate $\sin^{-1}\left[\frac{2^{x+1}.3^x}{1+(36)^x}\right]$ w.r.t x.

17. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$, prove that

$\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$, Where $-1 < x < 1$ and $-1 < y < 1$ [HINT: put $x^3 = \sin A$ and $y^3 = \sin B$]

18. If $f(x) = \sqrt{x^2+1}, g(x) = \frac{x+1}{x^2+1}$ and $h(x) = 2x - 3$ find $f'[h'(g'(x))]$.

19. If $x = \sec\theta - \cos\theta$ and $y = sec^n\theta - cos^n\theta$, then prove that $\frac{dy}{dx} = n\sqrt{\frac{y^2+4}{x^2+4}}$

20. If $x^y + y^x + x^x = m^n$, then find the value of $\frac{dy}{dx}$.

21. If $x = a\,cos^3\theta, y = a\,sin^3\theta$ then find $\frac{d^2y}{dx^2}$ at $x = \frac{\pi}{6}$

22. If $y = \tan^{-1}\left[\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right]$ where $0 < x < \frac{\pi}{2}$ find $\frac{dy}{dx}$

23. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then show that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

24. If $= \left[x + \sqrt{x^2+1}\right]^m$, show that $(x^2+1)y_2 + xy_1 - m^2y = 0$.

25. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log \quad)^2}$

26. If $y^{1/m} + y^{-1/m}$ = 2x then prove that $(x^2-1)y_2 + xy_1 = m^2y$.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ONE

1. If $y = \sin^2 x - \cos^2 x$, then $\frac{dy}{dx} =$

   (a) $2\sin x$ (b) $2\cos x$

   (c) $2\sin 2x$ (d) $-2\sin 2x$

2. The value of '4*k*' for which the function *f*(*x*) is continuous at *x* = 3.

   $$f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3}, & \text{when } x \neq 3 \\ 2k+1, & \text{when } x = 3 \end{cases}$$

   (a) 4 (b) 6

   (c) 11 (d) 22

3. Derivative of $\sin x$ with respect to $\cos x$ is

   (a) $\tan x$ (b) $-\tan x$

   (c) $\cot x$ (d) $-\cot x$

4. If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} =$

(a) n^2y (b) ny

(c) y (d) $-ny$

5. If $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ then $\dfrac{d^2y}{dx^2} =$

(a) $\dfrac{\sec^3\theta}{a}$ (b) $\dfrac{\sec^3\theta}{a\theta}$

(c) $\sec^3\theta$ (d) $\theta\sec^3\theta$

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ONE

1. A Function defined as

$$f(x) = \begin{cases} |x| - 3, & \text{when } x < 0 \\ 5 - |x|, & \text{when } x \geq 0 \end{cases}$$

is continuous on

(a) R (b) $R - \{0\}$

(c) $[0, \infty)$ (d) $(-\infty, 0]$

2. The function g(x) = $(\sin x + \cos x)$ is continuous at

(a) R (b) $R - \{0\}$

(c) $R - \left\{\dfrac{\rho}{2}\right\}$ (d) $R - \{\pi\}$

3. The value of the derivative of $|x-2| + |x-3|$ at $x = 2$ is

(a) 1 (b) 3

(c) 2 (d) 0

4. If $\sin y = x.\cos(a+y)$ then $\dfrac{dy}{dx}$?

(a) $\dfrac{\cos^2(a+y)}{\cos a}$ (b) $\dfrac{\cos^2(a+y)}{\sin a}$

(c) $\dfrac{\sin^2(a-y)}{\cos a}$ (d) $\dfrac{\sin^2(a+y)}{\sin a}$

5. If $y = \left(\dfrac{x^a}{x^b}\right)^{a-b} \cdot \left(\dfrac{x^b}{x^c}\right)^{b+c} \cdot \left(\dfrac{x^c}{x^a}\right)^{c+a}$, then $\dfrac{dy}{dx} =$

(a) 1 (b) abc

(c) $a + b + c$ (d) 0

ANSWERS

ONE MARK QUESTIONS

1. (d) 10
2. (d) $\frac{11}{4}$
3. (d) No value
4. (b) $\frac{\pi}{5}$
5. (a) $\frac{-2}{3}$
6. (c) $\frac{1}{2}$
7. (c) 3
8. (d) 1.5
9. (c) continuous and differentiable
10. (c) neither differentiable at x = 0 nor at x = 1
11. (b) $R-\left\{\frac{1}{2}\right\}$
12. (b) $\frac{-4x}{1-x^4}$
13. (c) $\frac{x}{\sqrt{1+x^2}}$
14. (b) 0
15. (a) $\frac{5}{x\log(x^5)\log(\log x^5)}$
16. (b) $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+x^2y=0$
17. (a) $\frac{\cos x}{2y-1}$
18. (a) $\frac{x}{\sqrt{1+x^2}}$

ASSERTION REASONING

18. Answer (d) Assertion (A) is false and Reason (R) is true
19. (b) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of the Assertion (A).
20. Solution (c) Assertion (A) is true and Reason (R) is false
21. Ans (a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A)
22. (a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A)

CASE BASED QUESTIONS

23. (a) f(x) = 2x – 5

    (b) f(x) = 1

    (c) Since the function is not continuous he will not get a pot.
24. (a) $\frac{1}{\sqrt{2}}$ (b) 2 (c) $\frac{16}{27}x^{11}$
25. (a) 9f' (3) + 6f(3) (b) 2f'(x)

TWO MARKS QUESTIONS

1. $\frac{2x\cos(x^2)}{\cos x\, e^{\sin x}}$
2. $\frac{y^2}{x[1-y\log x]}$
3. $x^x\,[1 + \log x] + 3x^2 + 3^x \log_e 3$
4. –7
5. $-\frac{1}{x^2}$
6. $\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$
7. $\frac{2x(x^2+2)}{|x^2+2|}$
8. $(-1,0) \cup (0,1)$
9. 0

11. $\text{Sin}\,(2\theta^2 + 2), \theta \neq 0$

12. 0

14. $10^{x^{10^x}}\, 10^x \log 10(1 - x\log 10)$

15. $x^x[1-\log x]$

16. -1

17. 1

18. $\frac{1}{x\log x}$

19. 2x $\sin x^4$

20. $\frac{1}{4\sqrt{x}\sqrt{1-x}\sqrt{\sin^{-1}\sqrt{x}}}$, where 0 < x < 1

THREE MARKS QUESTIONS

1. (I) Continuous (II) Discontinuous

   (III) Not Continuous at $x = 0$ (IV) Continuous

2. (I) K = -1 (II) K = $^1/_2$

   (III) K = 8

3. $a = 0,\ b = -1$

4. $a = \frac{-3}{2},\ b = R - \{0\},\ c = \frac{1}{2}$

5. $\lambda = -1$

6. $a = \frac{1}{2},\ b = 4$

7. $a = 3,\ b = 5$

8. 3a – 3b = 2

9. $-\frac{1}{2}$

10. $x^x\, x^{x^x}\left\{(1 + \log x)\log x + \frac{1}{x}\right\}$

11. $(x\cos x)^x[1 - x\ \tan x + (\log x\ \cos x)] + (x\sin x)^{1/x}\left[\frac{1+x\ cot\, x - log(x\ sin)^x}{x^2}\right]$

16. $\left[\frac{2^{x+1}3^x}{1+(36)^x}\right]\log 6$

18. $\frac{2}{\sqrt{5}}$

20. $\frac{dy}{dx} = \frac{x^x(1+\log x)+yx^{y-1}-y^x\log y}{x^y\log x + xy^{x-1}}$

21. $\frac{32}{27a}$

22. $-\frac{1}{2}$

SELF ASSESSMENT TEST-1

1. (C) 2. (C) 3. (D) 4. (A) 5. (B)

SELF ASSESSMENT TEST-2

1. (B) 2. (A) 3. (C) 4. (A) 5. (D)

CHAPTER 6

APPLICATION OF DERIVATIVES

The sight of soap bubble produced using a bubble wand is very exciting! One application of derivative is finding the rate of increase of size of the bubble (*dv*/*dt*) due to increasing radius, where *V* is the volume of spherical bubble and *r* is the radius. This can be calculated by knowing the rate of increase of radius with time (*dr*/*dt*).

APPLICATION OF DERIVATIVES

Topics to be covered as per C.B.S.E. revised syllabus (2025-26)

- Applications of derivatives:
- rate of change of quantities,
- increasing/decreasing functions,
- maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool).
- Simple problems (that illustrate basic principles and understanding of the subject as well as real life situations).

POINTS TO REMEMBER

- **Rate of change:** Let $y = f(x)$ be a function then the rate of change of y with respect to x is given by $\frac{dy}{dx} = f'(x)$ where a quantity y varies with another quantity x.

  $\left\{\frac{dy}{dx}\right\}_{x = x_1}$ or f' (x_1) represents the rate of change of y w.r.t. x at $x = x_1$.

- **Increasing and Decreasing Function**

  Let f be a real-valued function and let I be any interval in the domain of f. Then f is said to be

  a) Strictly increasing on I, if for all $x_{1,}\ x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

  b) Increasing on I, if for all $x_{1,}\ x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

  c) Strictly decreasing in I, if for all $x_{1,}\ x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

  d) Decreasing on I, if for all $x_{1,}\ x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

- **Derivative Test:** Let f be a continuous function on [a, b] and differentiable on (a, b). Then

  a) f is strictly increasing on [a, b] if $f'(x) > 0$ for each $x \in (a, b)$.

  b) f is increasing on [a, b] if $f'(x) \geq 0$ for each $x \in (a, b)$.

  c) f is strictly decreasing on [a, b] if $f'(x) < 0$ for each $x \in (a, b)$.

d) f is decreasing on [a, b] if $f'(x) \leq 0$ for each $x \in (a, b)$.

e) f is constant function on [a, b] if $f'(x) = 0$ for each $x \in (a, b)$.

- **Maxima and Minima**

a) Let f be a function and c be a point in the domain of f such that either f '(x)=0 or f '(x) does not exist are called critical points.

b) **First Derivative Test:** Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point *c* in interval I.

i. $f'(x)$ changes sign from positive to negative as x increases through *c*, then *c* is called the point of the local maxima.

ii. $f'(x)$ changes sign from negative to positive as *x* increases through *c*, then *c* is a point of *local minima*.

iii. $f'(x)$ does not change sign as *x* increases through *c*, then *c* is neither a point of *local maxima* nor a point of *local minima*. Such a point is called a point of *inflexion.*

c) **Second Derivative Test :** Let *f* be a function defined on an interval I and let $c \in I$. Let f be twice differentiable at c. Then

i. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value $f(c)$ is local maximum value of *f*.

ii. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. The value $f(c)$ is local minimum value of *f*.

iii. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

EXTREME VALUE OF A FUNCTION

Let $y = f(x)$ be a real function defined on an interval I and *C* be any point in I. Then *f* is said to have an extreme value in *I* if $f(c)$ is either maximum or minimum value of *f* in *I*.

Here, $f(c)$ is called the extreme value and *C* is called one of the extreme points.

Illustration:

Let $f(x) = (2x - 1)^2 + 3$.

Then, $f(x) \geq 3$, as $(2x - 1)^2 \geq 0$

For any real number 'x'

$\Rightarrow (2x - 1)^2 + 3 \geq 0 + 3$

Thus, minimum value of $f(x)$ is 3, which occurs at $x = \frac{1}{2}$

Also $f(x)$ has no maximum value as $f(x) \to \infty$ as $|x| \to \infty$

Illustration:

Let $g(x) = -(x - 1)^2 + 10$.

Then, $g(x) = 10 - (x - 1)^2 \leq 10 \;\; \forall x \in R$ as $(x - 1)^2$ is

Always greater them or equal to zero.

Thus maximum value of $g(x)$ is 10, which occurs at $x = 1$

Also $g(x)$ has no minimum value of $f(x) \to -\infty$ as $|x| \to \infty$.

Illustration:

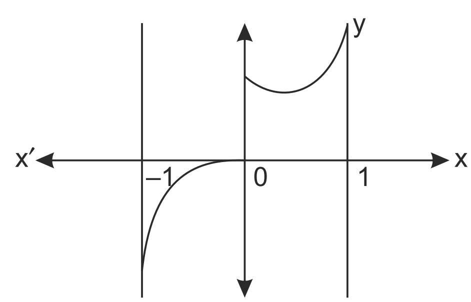
Neither maximum nor minimum value of a function.

Let as consider a function $f(x) = x^3$, $x \in (-1, 1)$

Since this function is an increasing function in (–1, 1), it should have minimum value at a point nearest to –1 and maximum value at a point nearest to 1.

But we can not locate such points (see figure)

So, $f(x) = x^3$, has neither maximum nor-minimum value in (–1, 1).



But, if we extend the domain of f to [–1, 1], then the function $f(x) = x^3$ has maximum value 1 at $x = 1$ and minimum value –1 at $x = -1$

Note: Every continuous function on an closed interval has a maximum and minimum

ONE MARK QUESTIONS

Multiple Choice Questions(MCQ)

1. If a function $f: R \to R$ is defined by $f(x) = 2x + \cos x$, then

   (a) f has a minimum at $x = \pi$

   (b) has a maximum at x=0

   (c) f is a decreasing function

   (d) f is an increasing function

2. If the radius of circle is increasing at the rate of 2cm/sec , then the area of circle when its radius is 20 cm is increasing at the rate of

   (a) $80\pi m^2/sec$ (b) $80\ m^2/sec$

   (c) $80\pi cm^2/sec$ (d) $80\ cm^2/sec$

3. The maximum value of $\frac{\log x}{x}$ is :

   (a) e (b) $2e$

   (c) $\frac{1}{e}$ (d) $\frac{2}{e}$

4. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is :

   (a) $[-l, \infty)$ (b) $(-\infty, -2]$

   (c) $[-2, -l)$ (d) $[-l, 1)$

5. The sides of an equilateral triangle are increasing at the rate of 2cm/sec. The rate at which its area increases, when its side is 10 cm is :

   (a) $10\ cm^2$ / sec (b) $10\sqrt{3}\ cm^2$ / sec

   (c) $\frac{10}{3} cm^2$ / sec (d) $\sqrt{3} cm^2$ / sec

6. The function f(x)= x^x, x > o is increasing on the interval

(a) (0, e] (b) (0, 1/e)

(c) [1/e, ∞) (d) None of these

7. The function f(x) = $2x^3 - 15x^2 + 36x + 6$ is increasing in the interval:

(a) (−∞, 2)U [3, ∞) (b) (−∞, 2)

(c) (−∞, 2]U [3, ∞) (d) [3, ∞)

8. A point on the curve $y^2 = 18x$ at which ordinate increases twice the rate of abscissa is :

(a) (2, 4) (b) (2, −4)

(c) $\left(\frac{-9}{8},\frac{9}{2}\right)$ (d) $\left(\frac{9}{8},\frac{9}{2}\right)$

9. The least value of function $f(x) = ax + \frac{b}{x}(x > 0, a > 0, b > 0)$ is:

(a) $\sqrt{ab}$ (b) $2\sqrt{ab}$

(c) ab (d) 2ab

10. At $x = \frac{5\pi}{6}$, the function f(x) = 2 sin 3x + 3cos 3x is

(a) Maximum (b) Minimum

(c) zero (d) Neither maximum nor minimum

11. The function tanx-x :

(a) always increases (b) always deccreases

(c) Remains contain (d) Sometime increases sometime decreases

12. The minimum value of $x^2 + \frac{250}{x}$ is:

(a) 75 (b) 55

(c) 50 (d) 20

13. In a sphere of radius r, a right circular cone of height having maximum curved surface area is inscribed. The expression for the square of curved surface of the cone is:

(a) $2\pi^2 rh(2rh + h^2)$ (b) $\pi^2 hr(2rh + h^2)$

(c) $2\pi^2 r(2rh^2 - h^3)$ (d) $2\pi^2 r^2(2rh - h^2)$

ASSERTION REASON TYPE QUESTIONS 1 Marks

Statement I is called Assertion (A) Statements II is called Reason R. Read the given statiments carefully and chose the correct answer from the four options given below.

(a) Both the statement are true and statement II is correct explantion of statement I

(b) Both the statmetns are treu and statement II is not the correct explanation of statement I.

(c) Statement I is true statement II is false

(d) Statement I is false and statement II is true

14. Statement I. The function f(x) = x^x, x > 0, is strictly increasing in $\left(\frac{1}{e}, \infty\right)$

Statement II : $\log_a x > b \Rightarrow x > a^b$ if a > 1

15. Let $a, b \in R$ be such that the function f given by $f(x) = \log|x| + bx^2 + an, x \neq 0$

has extreme values at x = – 1, and x = 2

Statement I : f has local maximum at x = – 1 and x = 2

Statement II : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$

16. Let $f(x) = 2x^3 - 15x^3 + 36x + 1$

Statement I : f is strictly decreasing in [2, 3]

Statement I :: f is strictly increasing in (– ∞, 2] U [3, ∞)

TWO MARKS QUESTIONS

1. The sum of the two numbers is 8, what will be the maximum value of the sum of their reciprocals.

2. Find the maximum value of $f(x) = 2x^3 - 24x + 107$ in the interval [1, 3]

3. If the rate of change of Area of a circle is equal to the rate of change its diameter. Find the radius of the circle.

4. The sides of on equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases, when side is 10 cm.

5. If there is an error of a% in measuring the edge of cube, then what is the percentge error in its surface?

6. If an error of k% is made in measuring the radius of a sphere, then what is the percentage error in its volume?

7. If the curves $y = 2e^x$ and $y = ae^{-x}$ intersect orthogonally, then find a.

8. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.

9. Prove that the function $f(x) = \tan x - 4x$ is strictly decreasing on $\left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$.

10. Find the point on the curve $y = x^2$, where the slope of the tangent is equal to the x co-ordinate of the point.

11. Use differentials to approximate the cube root of 66.

12. Find the maximum and minimum values of the function $f(x) = \sin(\sin x)$

13. Find the local maxima and minima of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$.

14. If $y = a \log x + bx^2 + x$ has its exteme values at $x = -1$ and $x = 2$, then find a and b.

15. If the radius of the circle increases from 5 into 5.1 cm, then find the increase in area.

THREE MARKS QUESTIONS

1. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of 900 cm^3 of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm.

2. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of $\frac{3}{2}$c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

3. The volume of a cube is increasing at a constant rate. Prove that the increase in its surface area varies inversely as the length of an edge of the cube.

4. A kite is moving horizontally at a height of 151.5 meters. If the speed of the kite is 10m/sec, how fast is the string being let out when the kite is 250 m away from the boy who is flying the kite ? The height of the boy is 1.5 m.

5. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 sec. and what is the average rate at which the water flows out during the first 5 seconds?

6. A man 2m tall, walk at a uniform speed of 6km/h away from a lamp post 6m high. Find the rate at which the length of his shadow increases.

7. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lower most. Its semi- vertical angle is $\tan^{-1}(0.5)$. water is poured into it at a constant rate of$5m^3/h$. Find the rate at which the level of the water is rising at the instant, when the depth of Water in the tank is 4m.

8. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate.

9. A conical vessel whose height is 10 meters and the radius of whose base is half that of the height is being filled with a liquid at a uniform rate of $1.5m^3/min$. find the rate at which the level of the water in the vessel is rising when it is 3m below the top of the vessel.

10. Let x and y be the sides of two squares such that$y = x - x^2$. Find the rate of change of area of the second square w.r.t. the area of the first square.

11. The length of a rectangle is increasing at the rate of 3.5 cm/sec. and its breadth is decreasing at the rate of 3 cm/sec. Find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm.

12. If the areas of a circle increases at a uniform rate, then prove that the perimeter various inversely as the radius.

13. Show that $f(x) = x^3 - 6x^2 + 18x + 5$ is an increasing function for all$x \in R$. Find its value when the rate of increase of f(x) is least.

    [Hint: Rate of increase is least when $f'(x)$ is least.]

14. Determine whether the following function is increasing or decreasing in the given interval: $f(x) = \cos\left(2x + \frac{\pi}{4}\right), \frac{3\pi}{8} \le x \le \frac{5\pi}{8}$.

15. Determine for which values of x, the function y=$x^4 - \frac{4x^3}{3}$ is increasing and for which it is decreasing.

16. Find the interval of increasing and decreasing of the function $f(x) = \frac{\log x}{x}$

17. Find the interval of increasing and decreasing of the function$f(x) = \sin x - \cos x,\ 0 < x < 2\pi$.

18. Show that $f(x) = x^2 e^{-x}, 0 \le x \le 2$ is increasing in the indicated interval.

19. Prove that the function $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

20. Find the intervals in which the following functionis decreasing.

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

21. Find the interval in which the function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$ is strictly decreasing.

22. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$,is strictly increasing the interval $\left(0, \frac{\pi}{4}\right)$.

23. Find the interval in which the function $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is increasing or decreasing.

24. Find the interval in which the function given by

$$f(x) = \frac{3x^4}{10} - \frac{4x^3}{5} - 3x^2 + \frac{36x}{5} + 11$$

(i) strictly increasing

(ii) strictly decreasing

25. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

26. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 Units/sec. then how fast is the slope of the curve changing when x=3?

27. If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

28. If the side of a cube be increased by 0.1%, find the corresponding increase in the volume of the cube.

29. Find the maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$.

30. Find the absolute maximum value and absolute minimum value of the following question $f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$ in [-2, 2.5]

31. Find the maximum and minimum values of $f(x) = x^{50} - x^{20}$ in the interval [0, 1]

32. Find the absolute maximum and absolute minimum value of $f(x) = (x-2)\sqrt{x-1}$ in [1, 9]

33. Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

FIVE MARKS QUESTIONS

1. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}\,r$.

2. If the sum of length of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

3. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

4. The sum of the surface areas of cuboids with sides x, 2x and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum if x = 3 radius of the sphere. Also find the minimum value of the sum of their volumes.

5. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

6. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ of the diameter of the sphere.

7. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

8. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle $\alpha \ is \ \frac{4}{27}\pi h^3 tan^2\alpha.$ Also show that height of the cylinder is $\frac{h}{3}$

9. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2,1).

10. Find the shortest distance between the line $y - x = 1$ and the curve $x = y^2$.

11. A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces, so that the combined area of the square and the circle is minimum?

12. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius r is $\frac{2r}{\sqrt{3}}$.

13. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ONE.

1. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of curve changing when $x = 3$

   (a) 72 units/sec (b) –72 units/sec

   (c) 54 units/sec (d) –54 units/sec

2. The function $f(x) = \tan x - 4x$, on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ is

   (a) strictly decreasing (b) strictly increasing

   (c) neither increasing nor deceasing

   (d) None of these

3. The curve $y = xe^x$ has minimum value equal to

(a) 1 (b) 0

(c) –e (d) / $\frac{1}{e}$

4. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate (in cm^2/sec) at which the area increases, when side is 10 cm is

(a) 10 (b) 5

(c) $10\sqrt{3}$ (d) $5\sqrt{3}$

5. If $ab = 2a + 3b$, $a > 0$, $b > 0$ then the minimum value of ab is

(a) 6 (b) 12

(c) 24 (d) 48

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ONE.

1. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$ where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a =

(a) 0 (b) 1

(c) 2 (d) 3

2. The interval in which $y = -x^3 + 3x^2 + 2022$ is increasing is

(a) $(-\infty, 0) \cup (2, \infty)$ (b) $(2, \infty)$

(c) $(0, 2)$ (d) $(-\infty, 0)$

3. The maximum value of the function $f(x) = 4\sin x.\cos x$ is

(a) 1 (b) 2

(c) 3 (d) 4

4. Which of the following function is decreasing on $\left(0, \frac{\pi}{2}\right)$

(a) $\cos x$ (b) $\sin x$

(c) $\tan x$ (d) $\sin 2x$

5. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. The rate at which the length of his shadow increases is

(a) 5 km/hr (b) 2 km/hr

(c) 3 km/hr (d) 2.5 km/hr

Answers

ONE MARK QUESTIONS

Answer

1. (d) f is an increasing
2. (c) $80\pi\ cm^2 / sec$
3. (c) $\frac{1}{e}$
4. (c) [–2, –1]
5. (b) $10\sqrt{3} cm^2 / sec$
6. (c) [1/e, ∞)
7. (c) (–∞, 2] U [3, ∞)
8. (d) $\left(\frac{9}{8}, \frac{7}{2}\right)$
9. (b) $2\sqrt{ab}$
10. (d) Neither maximum nor minimum
11. (a) always increases
12. (a) 75
13. (c) $2\pi^2 r(2rh^2 - h^3)$

ASSERTION REASON TYPE QUESTIONS

14. (a) Both the statements are true and statement II is correct explanation of statement I.
15. (a) Both the statements are true and statement II is correct explanation of statement I.
16. (b) Both the statements are true and statement II is not the correct explanation of statement I.

TWO MARKS QUESTIONS

1. $\frac{1}{2}$
2. 89
3. $\frac{1}{\pi}$ units
4. $10\sqrt{3}\ cm^2 / s$
5. 2a%
6. 3k %
7. ½
8. (2, 4)
10. (0, 0)
11. 4.042
12. sin 1, – sin 1
13. Local maxima at x = 1
    Local minima at x = 6
14. a = 2, b = –½
15. $\pi\ cm^2$

THREE MARKS QUESTIONS

1. $\frac{1}{\pi}\ cm / s$
2. $\frac{3}{8\pi}\ cm / min$
4. 8 m/sec.
5. 3000 L/s
6. 3 km/h
7. $\frac{35}{88}$ m/h
9. $\frac{6}{49\pi}$ m/min.

10. $1-3x+2x^2$

11. 8 cm^2/sec

13. 25

14. Increasing

15. Increasing for all $x \geq 1$

Decreasing for all $x \leq 1$

16. Increasing on (o, e)

Decreasing on [e, ∞)

17. Increasing on

$$\left(O, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$$

Decreasing on $[\frac{3\pi}{4} . \frac{7\pi}{4}]$

20. $(-\infty, 1] \text{U} [2, 3]$

21. [1,∞]

23. increasing on [0, ∞)

Decreasing $(-\infty, 0]$

24. (i) Strictly increasing

[-2,1] U [3, ∞)

(ii) Strictly decreasing

(-∞, -2] U [1, 3]

26. decrease 72 units/sec.

27. $\pi\ cm^2$

28. 0.3%

29. max. value $=\frac{3}{4}$, mim value $=\frac{1}{2}$

30. ab. Max.= $\frac{157}{8}$, ab. Min. = $\frac{-7}{4}$

31. max.value=0,

min.value = $\frac{-3}{5}\left[\frac{2}{5}\right]^{2/3}$

32. ab. Max = 14 at x = 9

ab. Min.= $\frac{-3}{4^{4/3}}$ at $x = \frac{5}{4}$

33. π

FIVE MARKS QUESTIONS

4. $18r^3 + \frac{4}{3}\pi r^3$

9. (1, 2)

10. $\frac{3\sqrt{2}}{8}$

11. $\frac{144}{\pi+4}m, \frac{36\pi}{\pi+4}m$

13. 2ab sq. Units.

SELF ASSESSMENT TEST-1

1. (b) 2. (a) 3. (d) 4. (c) 5. (c)

SELF ASSESSMENT TEST-2

1. (c) 2. (c) 3. (b) 4. (a) 5. (d)

CHAPTER 7

INTEGRALS

There are many applications of integration in the field such as Physics, Engineering, Business, Economics etc. One of the important application of integration is finding the profit function of producing a certain number of cars if the marginal cost and revenue function are known. Companies can thus determine the maximum profit that can be earned and in this way plan their production, labour and other infrastructure accordingly.

INTEGRALS

Topics to be covered as per C.B.S.E. revised syllabus (2025-26)

- Integration as inverse process of differentiation
- Integration of a variety of functions by substitution, by partial fractions and by parts
- Evaluation of simple integrals of the following types and problems based on them.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{px + q}{ax^2 \pm bx + c} dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx, \int \sqrt{a^2 \pm x^2} dx, \int \sqrt{x^2 - a^2}\, dx$$

$$\int \sqrt{ax^2 + bx + c}\, dx$$

Fundamental Theorem of Calculus (without proof).

- Basic properties of definite integrals and evaluation of definite integrals.

POINTS TO REMEMBER

- Integration or anti derivative is the reverse process of Differentiation.
- Let $\frac{d}{dx}F(x) = f(x)$ then we write $\int f(x)\,dx = F(x) + c$.
- These integrals are called indefinite integrals and c is called constant of integration.
- From geometrical point of view, an indefinite integral is the collection of family of curves each of which is obtained by translating one of the curves parallel to itself upwards or downwards along y-axis.

STANDARD FORMULAE

1. $\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + c, & n \neq -1 \\ \log_e|x| + c, & n = -1 \end{cases}$

2. $\int (ax+b)^n\,dx = \begin{cases} \frac{(ax+b)^{n+1}}{(n+1)a} + c, & n \neq -1 \\ \frac{1}{a}\log|ax+b| + c, & n = -1 \end{cases}$

3. $\int \sin x\,dx = -\cos x + c.$

4. $\int \cos x\,dx = \sin x + c$

5. $\int \tan x.\,dx = -\log|\cos x| + c = \log|\sec x| + c.$

6. $\int \cot x\,dx = \log|\sin x| + c.$

7. $\int \sec^2 x\,dx = \tan x + c$

8. $\int \text{cosec}^2\,x\,dx = -\cot x + c$

9. $\int \sec x\,\tan x\,dx = \sec x + c$

10. $\int \text{cosec}\,x\,\cot x\,dx = -\text{cosec}\,x + c$

11. $\int \sec x \, dx = \log|\sec x + \tan x| + c$

$$= \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$$

12. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

$$= \log\left|\tan\frac{x}{2}\right| + c$$

13. $\int e^x \, dx = e^x + c$

14. $\int a^x \, dx = \frac{a^x}{\log a} + c$

15. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c \, , |x| < 1$

$$= -\cos^{-1} x + c$$

16. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$$= -\cot^{-1} x + c$$

17. $\int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \sec^{-1} x + c \, , |x| > 1$

$$= -\operatorname{cosec}^{-1} x + c$$

18. $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

19. $\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

20. $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

21. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c$

22. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c$

23. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

24. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

25. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + c$

26. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

RULES OF INTEGRATION

1. $\int [(f_1(x) \pm f_2(x) \pm \ldots\ldots \pm f_x(x)]\, dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \ldots\ldots \pm \int f_x(x)dx$

2. $\int k.f(x)dx = k\int f(x)dx.$

3. $\int e^x \{f(x) + f'(x)\}dx = e^x f(x) + c$

INTEGRATION BY SUBSTITUTION

1. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

2. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$

3. $\int \frac{f'(x)}{[f(x)]^n} dx = \frac{(f(x))^{-n+1}}{-n+1} + c$

INTEGRATION BY PARTS

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int\left[f'(x)\int g(x)\, dx\right]$$

DEFINITE INTEGRALS

$$\int_a^b f(x)dx = F(b) - F(a), \text{where } F(x) = \int f(x)\, dx$$

DEFINITE INTEGRAL AS A LIMIT OF SUMS.

$$\int_a^b f(x)dx = \lim_{h \to o} h\,[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+n-1h)]$$

$Where\ h = \frac{b-a}{h}$ $or \int_a^b f(x)dx = \lim_{h \to o}[h \sum_{r=1}^{n} f(a+rh)]$

PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f(x) = -\int_b^a f(x)dx$

2. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

4. (i) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

(ii) $\int_o^a f(x)\,dx = \int_o^a f(a-x)dx$

5. $\int_{-a}^{a} f(x)dx = 2\int_o^a f(x)dx,$ $\quad if\ f\ (x)\ is\ even\ function$

6. $\int_{-a}^{a} f\ (x)dx = 0\ if\ f\ (x)\ is\ an\ odd\ function$

7. $\int_0^{2a} f\ (x)dx = \begin{cases} 2\int_0^a f\ (x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Illustration:

Evaluate $\int e^x \left(\frac{x-2}{x+4}\right)^2 dx$

Solution: $I = \int e^x \left(\frac{x-2}{x+4}\right)^2 dx = \int e^x \left(1-\frac{2}{x+4}\right)^2 dx$

$= \int e^x \left[\left(1-\frac{4}{x+4}\right)+\frac{4}{(x+4)^2}\right]dx$ It is of the form $e^x \left[f(x) + f'(x)\right]$ dx

$= \int e^x \left[f(x)+f'(x)\right]dx,$ where $f(x) = 1-\frac{4}{x+4}$

$= e^x f(x) + C = e^x\left(1-\frac{4}{x+4}\right)+C = \frac{xe^x}{x+4}+C$

Illustration:

Find $\int \frac{x^2+1}{(x+1)^2}dx$

Solution: $\int \frac{x^2+1}{(x+1)^2}dx = \int \frac{(x+1)^2-2x}{(x+1)^2}dx$

$= \int \frac{(x+1)^2-2(x+1)+2}{(x+1)^2}dx$

$= \int \left[1-\frac{2}{x+1}+\frac{2}{(x+1)^2}\right]dx$

$= x-2\log|x+1|-\frac{2}{x+1}+C$

Illustration:

Evaluate $\int_0^{\pi/4} \frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx$

Solution: $\int_0^{\pi/4} \frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx = \int_0^{\pi/4} \frac{\tan^2 x\sec^2 x}{(\tan^3 x+1)^2}dx$

[dividing Num and Den by $\cos^6 x$]

Put $z = \tan^3 x + 1$,

then $dz = 3\tan^2 x\sec^2 x\, dx$

Also when $x = 0$, $z = 0$ and when $x = \frac{\pi}{4}$, $z = 2$

Now $I = \frac{1}{3}\int_{2}^{1}\frac{dz}{z^2} = -\frac{1}{3}\left[\frac{1}{z}\right]_1^2 = -\frac{1}{3}\left[\frac{1}{2}-1\right] = \frac{1}{6}$

Illustration:

Find $\int_{-\pi/4}^{\pi/4}\frac{x+\pi/4}{2-\cos 2x}dx$

Solution: $\int_{-\pi/4}^{\pi/4}\frac{x+\frac{\pi}{4}}{2-\cos 2x}dx = \int_{-\pi/4}^{\pi/4}\frac{x}{2-\cos 2x}dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4}\frac{1}{2-\cos 2x}dx$

$= 0 + \frac{\pi}{4}.2\int_{0}^{\pi/4}\frac{dx}{2-\cos x}$ [Since first functionis an even function and second function is an odd function

$= \frac{\pi}{2}\int_{0}^{\pi/4}\frac{dx}{2(1-2\sin^2 x)}$

$= \frac{\pi}{2}\int_{0}^{\pi/4}\frac{dx}{2\sin^2 x+1}$

$= \frac{\pi}{2}\int_{0}^{\pi/4}\frac{\sec^2 x}{3\tan^2 x+1}dx$ [dividing num and den by $\cos^2 x$]

Put $z = \sqrt{3}\tan x$, then $dz = \sqrt{3}\sec^2 x\,dx$

Also when $x = 0,\ z = 0,$ and when $x = \frac{\pi}{4},\ z = \sqrt{3}$

$\therefore$ From (i), $I = \frac{\pi}{2}\frac{1}{\sqrt{3}}\int_{0}^{\sqrt{3}}\frac{dz}{z^2+1} = \frac{\pi}{2\sqrt{3}}\left[\tan^{-1} z\right]_0^{\sqrt{3}}$

$= \frac{\pi}{2\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}0\right]$

$= \frac{\pi}{2\sqrt{3}}\tan^{-1}\sqrt{3}$

$= \frac{\pi}{2\sqrt{3}}.\frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$

ONE MARK QUESTIONS

Evaluate the following integrals:

1. Integrate $\int_0^2 (x^2 + x + 1)dx$

(a) $\frac{15}{2}$ (b) 20/5

(c) 20/3 (d) 3/20

2. $\int_0^\pi \sin^2 x \, dx =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) 2π (d) 4π

3. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ equal to:

(a) $-\frac{1}{\sin x + \cos x} + c$ (b) $\log |\sin x + \cos x| + c$

(c) $\frac{1}{(\sin x + \cos x)^2}$ (d) $\log |\sin x - \cos x| + c$

4. $\int \frac{(1+\log x)^2}{1+x^2} dx$ is :

(a) $\frac{1}{3}(1+\log x)^3 + c$ (b) $\frac{1}{2}(1+\log x)^2 + c$

(c) $\log(\log 1 + x) + c^2$ (d) None of these

5. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x . \cos^2 x} dx$ is equal to

(a) $\tan x + \cos x + c$ (b) $\tan x + \text{cosec } x + c$

(c) $\tan x + \cot x + c$ (d) $\tan x + \sec x + c$

6. The value of $\int_{\pi/6}^{\pi/3} \frac{dx}{\sin 2x}$ is :

(a) $\frac{1}{2}\log(-1)$ (b) $\log(-1)$

(c) log3 (d) $\log\sqrt{3}$

7. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is:

(a) 1 (b) 0

(c) -1 (d) $\frac{\pi}{4}$

8. $\int \frac{x^9}{(4x^2+1)^6}dx$ is equal to

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+c$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+c$

(c) $\frac{1}{10x}\left(\frac{1+4}{x^2}\right)^{-5}+c$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+c$

9. If $\int \frac{x^3}{\sqrt{1+x^2}}dx = 9(1+x^2)^{3/2} + b\sqrt{1+x^c} + c$

(a) $a=\frac{1}{3}, b=1$ (b) $a=-\frac{1}{3}, b=1$

(c) $a=-\frac{1}{3}, b=-1$ (d) $a=\frac{1}{3}, b=-1$

ASSERTION-REASON BASED QUESTIONS

In the following questions, a statement of ssertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A

(b) Both A and R are true but R is not the correct explanation of A

(c) A is true but R is false

(d) A is false but R is true.

1. Assertion (A) : $\int \frac{dx}{x^2+2x+3} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$

   Reason (R) : $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

2. Assertion (A) : $\int e^x [\sin x - \cos x] dx = e^x \sin x + c$

   Reason (R) : $\int e^x [f(x) + f'(x)] dx = e^x (f(x) + c$

3. Assertion (A) : $\int_{-2}^{2} \log\left(\frac{1+x}{1-x}\right) dx = 0$

   Reason (R) : $\int_0^{2a} f(x) dx = 0 \; if \; f(2a-x)$

4. Assertion (A) : $\int_{\pi/6}^{\pi/3} \frac{1}{1+(\tan x)^{11/5}} dx = \frac{\pi}{12}$

   Reason (R) : $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

TWO MARKS QUESTIONS

Evaluate :

1. $\int e^{[\log(x+1)-\log x]} dx$

2. $\int \frac{1}{\sqrt{x+1}+\sqrt{x+2}} dx$

3. $\int \sin x \sin 2x\, dx$

4. $\int \left[\frac{x}{a}+\frac{a}{x}+x^a+a^x\right] dx$

5. $\int_0^{\pi/2} \log\left(\frac{5+3\cos x}{5+3\sin x}\right) dx$

6. $\int \frac{a^x+b^x}{c^x} dx$

7. $\int \left(\sqrt{ax}-\frac{1}{\sqrt{ax}}\right)^2 dx$

8. $\int e^x 2^x dx$

9. $\int 2^{2^{2^x}} 2^{2^x} 2^x dx$

10. $\int \frac{\sin\left(2\tan^{-1} x\right)}{1+x^2} dx$

11. $\int x \log 2x\, dx$

12. $\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$

13. $\int_0^{\pi/2} e^x (\sin x - \cos x) dx$

14. $\int_4^9 \frac{\sqrt{x}}{\left(30-x^{3/2}\right)^2} dx$

15. $\int_0^1 \frac{dx}{e^x+e^{-x}} dx$

16. $\int \frac{\log|\sin x|}{\tan x} dx$

17. $\int \frac{\sin^4 x+\cos^4 x}{\sin^3 x+\cos^3 x} dx$

18. $\int \sqrt{\tan x}\,(1+\tan^2 x) dx$

19. $\int \frac{\sin 2x}{(a+b\cos x)^2} dx$

20. $\int \frac{x^2-x+2}{x^2+1} dx$

THREE MARKS QUESTIONS

Evaluate :

1. (i) $\int \frac{x\, cosec\, (\tan^{-1} x^2)}{1+x^4}\, dx$

(ii) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\, dx$

(iii) $\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$

(iv) $\int \frac{\cos(x+a)}{\cos(x-a)}\, dx$

(v) $\int \cos 2x\ \cos 4x\ \cos 6x\ dx$

(vi) $\int \tan 2x \tan 3x \tan 5x\, dx$

(vii) $\int \sin^2 x \cos^4 x\ dx$

(viii) $\int \cot^3 x\, cosec^4 x\, dx$

(ix) $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx$ [Hint: Put $a^2 \sin^2 x + b^2 \cos^2 x = t \text{ or } t^2$

(x) $\int \frac{1}{\sqrt{\cos^3 x\ \cos(x+a)}}\, dx$

(xi) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$

(xii) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

Evaluate :

2. (i) $\int \frac{x}{x^4+x^2+1} dx$

(ii) $\int \frac{1}{x[6(\log x)^2+7 \log x+2]} dx$

(iii) $\int \frac{1}{\sqrt{\sin^3 x \cos^5 x}} dx$

(iv) $\int \frac{x^2+1}{x^4+1} dx$

(v) $\int \frac{1}{\sqrt{(x-a)(x-b)}} dx$

(vi) $\int \frac{5x-2}{3x^2+2x+1} dx$

(vii) $\int \frac{x^2}{x^2+6x+1} dx$

(viii) $\int \frac{x+2}{\sqrt{4x-x^2}} dx$

(ix) $\int x \sqrt{1+x-x^2} dx$

(x) $\int \frac{\sin^4 x}{\cos^8 x} dx$

(xi) $\int \sqrt{\sec x - 1} dx$ [Hint: Multiply and divided by $\sqrt{\sec x + 1}$]

Evaluate :

3. (i) $\int \frac{dx}{x(x^7+1)}$

(ii) $\int \frac{3x+5}{x^3-x^2-x+1} dx$

(iii) $\int \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos\theta - 2}\, d\theta$

(iv) $\int \frac{dx}{(2-x)(x^2+3)}$

(v) $\int \frac{x^2+x+2}{(x-2)(x-1)}\, dx$

(vi) $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\, dx$

(vii) $\int \frac{dx}{(2x+1)(x^2+4)}$

(viii) $\int \frac{x^2-1}{x^4+x^2+1}\, dx$

(ix) $\int \sqrt{\tan x}\, dx$

(x) $\int \frac{dx}{\sin x - \sin 2x}$

4. Evaluate:

(i) $\int x^5 \sin x^3\, dx$

(ii) $\int \sec^3 x\, dx$

(iii) $\int e^{ax} \cos(bx + c)\, dx$

(iv) $\int \sin^{-1}\left(\frac{6x}{1+9x^2}\right) dx$ [Hint: Put $3x = \tan\theta$]

(v) $\int \cos\sqrt{x}\, dx$

(vi) $\int x^3 \tan^{-1} x\, dx$

(vii) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$

(viii) $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$

(ix) $\int \sqrt{2ax - x^2}\, dx$

(x) $\int e^x \frac{(x^2+1)}{(x+1)^2} dx$

(xi) $\int x^3 \sin^{-1}\left(\frac{1}{x}\right) dx$

(xii) $\int \left\{\log(\log x) + \frac{1}{(\log x)^2}\right\} dx$ [Hint: Put $\begin{matrix}\log x = t \\ x = e^t\end{matrix}$]

(xiii) $\int (6x+5)\sqrt{6 + x - x^2}\, dx$

(xiv) $\int \frac{1}{x^3+1} dx$

(xv) $\int \tan^{-1}\left(\frac{x-5}{1+5x}\right) dx$

(xvi) $\int \frac{dx}{5+4\cos x}$

5. Evaluate the following definite integrals:

(i) $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\ \sin 2x}\ dx$

(ii) $\int_0^{\pi/2} \cos 2x\ \log \sin x\, dx$

(iii) $\int_0^1 x \sqrt{\frac{1-x^2}{1+x^2}} dx$

(iv) $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\ dx$

(v) $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$

(vi) $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right) dx$

(vii) $\int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x}\, dx$

(viii) $\int_0^1 x \log\left(1 + \frac{x}{2}\right) dx$

(ix) $\int_{-1}^{1/2} |x \cos \pi x|\, dx$

(x) $\int_{-\pi}^{\pi} (\cos a x - \sin b x)^2\, dx$

6. Evaluate:

(i) $\int_2^5 [|x-2| + |x-3| + |x-4|]\, dx$

(ii) $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

(iii) $\int_{-1}^{1} e^{\tan^{-1} x} \left[\frac{1+x+x^2}{1+x^2}\right] dx$

(iv) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$

(v) $\int_0^2 [x^2]\, dx$

(vi) $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$

(vii) $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x}\, dx$ [*Hint: use* $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

7. Evaluate the following integrals:

(i) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

(ii) $\int_{-\pi/2}^{\pi/2} \left(\sin|x| + \cos|x|\right) dx$

(iii) $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$

(iv) $\int_0^{\pi} \frac{x \tan x}{\sec x + \operatorname{cosec} x}\, dx$

(v) $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$

8. Evaluate

(*i*) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx \quad x \in [0, 1]$

(*ii*) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$

(iii) $\int \frac{x^2 e^x}{(x+2)^2} dx$

(iv) $\int \frac{x^2}{(x \sin x + \cos x)^2}\, dx$

(v) $\int \sin^{-1} \sqrt{\frac{x}{a+x}}\, dx$

(vi) $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin\ 2x}}\ dx$

(vii) $\int \frac{\sin x}{\sin 4x}\ dx$

(viii) $\int_{-1}^{3/2} |x\ \sin \pi x|\ dx$

(ix) $\int \frac{\sin(x-a)}{\sin(x+a)}\ dx$

(x) $\int \frac{x^2}{(x^2+4)(x^2+9)}\ dx$

(xi) $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\ dx$

FIVE MARKS QUESTIONS

9. Evaluate the following integrals:

(i) $\int \frac{x^5+4}{x^5-x}\ dx$

(ii) $\int \frac{2e^t}{e^{3t} - 6e^{2t} + 11e^t - 6}\ dt$

(iii) $\int \frac{2x^3}{(x+1)(x-3)^2}\ dx$

(iv) $\int \frac{1+\sin x}{\sin x\,(1+\cos x)}\ dx$

(v) $\int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

(vi) $\int_{0}^{1} x\sqrt{\frac{1-x^2}{1+x^2}}dx$

(vii) $\int_{0}^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\ dx$

10. Evaluate the following integrals as limit of sums:

(i) $\int_{2}^{4} (2x+1)\ dx$

(ii) $\int_{0}^{2} (x^2+3)\ dx$

(iii) $\int_{1}^{3} (3x^2-2x+4)\ dx$

(iv) $\int_{0}^{4} (3x^2+e^{2x})\ dx$

(v) $\int_{0}^{1} e^{2-3x}\ dx$

(vi) $\int_{0}^{1} (3x^2+2x+1)\ dx$

11. Evaluate:

(i) $\int \frac{dx}{(\sin x-2\cos x)(2\sin x+\cos x)}$

(ii) $\int_{0}^{1} \frac{\log(1+x)}{1+x^2}\ dx$

(iii) $\int_{0}^{\pi/2} (2\log\sin x-\log\sin 2x)\ dx$

12. $\int_0^1 x(\tan^{-1} x)^2 \, dx$

13. $\int_0^{\pi/2} \log \sin x \, dx$

14. $Prove\ that \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2 \int_0^1 \tan^{-1} x \, dx$

Hence or otherwise evaluate the integral $\int \tan^{-1}(1-x+x^2)\, dx$.

15. $Evaluate \int_0^{\pi/2} \frac{sin^2 x}{\sin x+\cos x} dx.$

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. $\mathrm{I} = \int (x^8+1)(x^4+1)(x^2+1)(x+1)(x-1)dx =$

(a) $x^{16} - 1 + c$ (b) $x^{17} - x + c$

(c) $\frac{x^{17}}{17} - x + c$ (d) $\frac{x^{16}}{16} - x + c$

2. $\int \sin(x^2+2022).d(x^2) =$

(a) $2x.\sin(x^2+2022)+c$ (b) $-2x.\cos(x^2+2022)+c$

(c) $\sin(x^2+2022)+c$ (d) $-\cos(x^2+2022)+c$

3. $\int \cos^3 x.\sqrt{\sin x}\, dx = \frac{2\sin^a x}{3} - \frac{2\sin^b x}{7} + c$, then $(a+b) =$

(a) 2 (b) 4

(c) 5 (d) 6

4. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x.\cos^2 x} dx =$

(a) $\tan x + \cot x + c$ (b) $-\tan x + \cot x + c$

(c) tanx + cosecx + c (d) tanx + secx + c

5. $\int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{k}$, then $k =$

(a) 0.25 (b) 0.5

(c) 1 (d) 4

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. $\int_0^{\pi/2} \log \tan x\, dx =$

(a) 0 (b) 1

(c) π (d) $\frac{\pi}{2}$

2. $\int_0^{\pi} \frac{x}{1+\sin x} dx =$

(a) 4π (b) $\frac{\pi}{2}$

(c) π (d) 2π

3. $\int \log(x^2+1) dx =$

(a) $x\log(x^2+1) - 2x + 2\tan^{-1} x + c$ (b) $x\log(x^2+1) - 2x - 2\tan^{-1} x + c$

(c) $x\log(x^2+1) + 2x + 2\tan^{-1} x + c$ (d) None of these

4. $\int e^x . \sin x\, dx =$

(a) $\frac{e^x(\sin x + \cos x)}{2} + c$ (b) $\frac{e^x(\sin x - \cos x)}{2} + c$

(c) $\frac{e^x(-\sin x + \cos x)}{2} + c$ (d) $\frac{-e^x(\sin x - \cos x)}{2} + c$

5. $\int \cos^2 x\, dx = ax + b\sin 2x + c$, then $(2a + 4b + 1) =$

(a) 0 (b) 1

(c) 3 (d) –7

Answers

ONE MARKS QUESTIONS

1. (c) $\frac{20}{3}$
2. (a) $\frac{\pi}{2}$
3. (b) $\log|\sin x + \cos x| + c$
4. (a) $\frac{1}{3}(1+\log x)^3 + c$
5. (c) tanx + cot x + c
6. (c) log 3
7. (b) 0
8. $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5} + c$
9. $a = \frac{1}{3}, b = -1$

INTEGRAL ASSERTION REASONS

1. A is true and R is correct explanation of A
2. Option (d) is correct
3. Option (b) is correct
4. (a) A is true and R is correct explanation of A

TWO MARKS QUESTIONS

1. $x + \log x + c$
2. $\frac{2}{3}\left[(x+2)^{3/2} - (x+1)^{3/2}\right] + c$
3. $\frac{-1}{2}\left[\frac{\sin 3x}{3} - \sin x\right] + c$
4. $\frac{1}{a}\frac{x^2}{2} + a\log|x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$
5. 0
6. $\frac{\left(\frac{a}{c}\right)^x}{\log\left|\frac{a}{c}\right|} + \frac{\left(\frac{b}{c}\right)^x}{\log\left|\frac{b}{c}\right|} + c$
7. $\frac{ax^2}{2} + \frac{\log|x|}{a} - 2x + c$
8. $\frac{2^x e^x}{\log(2e)} + c$
9. $\frac{2^{2^{2^x}}}{(\log 2)^3} + C$
10. $\frac{-\left[\cos\left(2\tan^{-1}x\right)\right]}{2} + C$
11. $\frac{x^2}{2}\log 2x - \frac{x^2}{4} + C$
12. 1
13. 1
14. $\frac{19}{99}$
15. $\tan^{-1} e - \frac{\pi}{4}$
16. $\frac{\log|\sin x|^2}{2} - C$
17. log|secx + tan x|
    + log |cosec x – cot x| + C
18. $\frac{2}{3}(\tan x)^{3/2} + C$
19. $-\frac{2}{b^2}\left[\log|a + b\cos x| + \frac{a}{a + b\cos x}\right] + C$
20. $x - \frac{1}{2}\log\left|x^2 + 1\right| + \tan^{-1}x + C$

THREE MARKS QUESTIONS

1. (i) $\frac{1}{2}\log\left[\operatorname{cosec}(\tan^{-1}x^2) - \frac{1}{x^2}\right] + c$

(ii) $\frac{1}{2}\left(x^2 - x\sqrt{x^2-1}\right) + \frac{1}{2}\log\left|x + \sqrt{x^2-1}\right| + c$

(iii) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c$

(iv) $x\cos 2a - \sin 2a \ \log|\sec(x-a)| + c$

(v) $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

(vi) $\frac{1}{5}\log|\sec 5x| - \frac{1}{2}\log|\sec 2x| - \frac{1}{3}\log|\sec 3x| + c$

(vii) $\frac{1}{32}\left[2x + \frac{1}{2}\sin 2x - \frac{1}{2}\sin 4x - \frac{1}{6}\sin 6x\right] + c$

(viii) $-\left(\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4}\right) + c$

(ix) $\frac{1}{a^2-b^2}\sqrt{a^2\sin^2 x + b^2\cos^2 x} + c$

(x) $-2\operatorname{cosec} a\sqrt{\cos a - \tan x \sin a} + c$

(xii) $\tan x - \cot x - 3x + c$

(vi) $\sin^{-1}[\sin x - \cos x] + c$

2. (i) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$

(ii) $\log\left|\frac{2\log x}{3\log x}\right| + c$

(iii) $\frac{-2}{\sqrt{\tan x}} + \frac{2}{3}\tan^{3/2}x + c$

(iv) $\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right\}+c$

(v) $2\log\left|\sqrt{x-a}+\sqrt{x-b}\right|+c$

(vi) $\frac{5}{6}\log|3x^2+2x+1|+\frac{-11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+c$

(vii) $x-3\log|x^2+6x+12|+2\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right)+c$

(viii) $-\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+c$

(ix) $-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$

(x) $\frac{\tan^5 x}{5}+\frac{\tan^7 x}{7}+c$

(xi) $-\log\left|\cos x+\frac{1}{2}+\sqrt{\cos^2 x+\cos x}\right|+c$

3. (i) $\frac{1}{7}\log\left|\frac{x^7}{x^7+1}\right|+c$

(ii) $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+c$

(iii) $\frac{-2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|1+\cos\theta|+c$

(iv) $\frac{1}{14}\log\left|\frac{x^2+3}{(2-x)^2}\right|+\frac{2}{7\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$

(v) $x+4\log\left|\frac{(x-2)^2}{x-1}\right|+c$

(vi) $x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+c$

(vii) $\frac{2}{17}\log|2x+1| - \frac{1}{17}\log|x^2+4| + \frac{1}{34}\tan^{-1}\frac{x}{2} + c$

(viii) $\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$

(ix) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}}\ \log\left|\frac{\tan x - \sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right| + c$

(x) $-\frac{1}{2}\log|\cos x - 1| - \frac{1}{6}\log|\cos x + 1| + \frac{2}{3}\log|1 - 2\cos x| + c$

4. (i) $\frac{1}{3}[-x^3\cos x^3 + \sin x^3] + c$

(ii) $\frac{1}{2}[\sec x\tan x + \log|\sec x + \tan x|] + c$

(iii) $\frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + c$

(iv) $2x\ \tan^{-1} 3x - \frac{1}{3}\log|1+9x^2| + c$

(v) $2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + c$

(vi) $\left(\frac{x^4-1}{4}\right)\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} + c$

(vii) $\frac{1}{2}e^{2x}\tan x + c$

(viii) $\frac{x}{\log x} + c$

(ix) $\left(\frac{x-a}{2}\right)\sqrt{2ax - x^x} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$

(x) $e^x\left(\frac{x-1}{x+1}\right) + c$

(xi) $\frac{x^4}{4}\sin^{-1}\left(\frac{1}{x}\right) + \frac{x^2+2}{12}\sqrt{x^2-1} + c$

(xii) $x\log|\log x| - \frac{x}{\log x} + c$

(xiii) $-2(6+x-x^2)^{\frac{3}{2}} + 8\left[\frac{2x-1}{4}\sqrt{6+x-x^2} + \frac{25}{8}\sin^{-1}\left(\frac{2x-1}{5}\right)\right] + c$

(xiv) $\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

(xv) $x\tan^{-1}x-\frac{1}{2}\log|1+x^2|-x\tan^{-1}5+c$

(xvi) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$

5. (i) $\frac{1}{20}\log 3$

(ii) $-\pi/4$

(iii) $\frac{\pi}{4}-\frac{1}{2}$

(iv) $\frac{\pi}{4}-\frac{1}{2}\log 2$

(v) $\frac{\pi}{2}$

(vi) $\pi/4$

(vii) $\pi/2$

(viii) $\frac{3}{4}+\frac{3}{2}\log\frac{2}{3}$

(ix) $\frac{3}{2\pi}-\frac{1}{\pi^2}$

(x) $2\pi+\frac{1}{2a}\sin 2a\pi-\frac{1}{2b}\sin 2b\pi$

6. (i) $\frac{1}{2}$

(ii) π

(iii) $e^{\pi/4}+e^{-\pi/4}$

(iv) $\frac{1}{4}\pi^2$

(v) $5-\sqrt{3}-\sqrt{2}$

(vi) $\frac{\pi^2}{16}$ (vii) $\frac{\pi^2}{2a}$

7. (i) $\frac{\pi}{12}$ (ii) 2

(iii) $\frac{\pi}{2}$ (iv) $\frac{\pi^2}{4}$

(v) aπ

8. (i) $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2\sqrt{x-x^2}}{\pi} - x + c$

(ii) $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + c$

(iii) $\frac{x-2}{x+2}e^x + c$

(iv) $\frac{\sin x - x\cos x}{x\sin x + \cos x} + c$

(v) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$

(vi) $2\sin^{-1}\frac{\sqrt{3}-1}{2}$

(vii) $\frac{1}{8}\log\left|\frac{1-\sin x}{1+\sin x}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + c$

(viii) $\frac{3}{\pi} + \frac{1}{\pi^2}$

(ix) $(\cos 2a)(x+a) - (\sin 2a)\log|\sin(x+a)| + c$

(x) $-\frac{4}{5}\log|x^2+4| + \frac{9}{5}\log|x^2+9| + c$

(xi) $-\left(\frac{1}{2}\sin 2x + \sin x\right) + c$

9. (i) $x - 4\log|x| + \frac{5}{4}\log|x-1| + \frac{3}{4}\log|x+1| + \log|x^2+1|$

(ii) $\frac{-1}{2}\tan^{-1}x + c$

(iii) $\log\left|\frac{(e^t-1)(e^t-3)}{(e^t-2)^2}\right| + c$

(iv) $2x-\frac{1}{8}\log|x+1|+\frac{81}{8}\log|x-3|-\frac{27}{2(x-3)}+c$

(v) $\frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos}\right|+\frac{1}{2(1+\cos\ )}+\tan\frac{x}{2}+c$

(vi) $\frac{\pi}{\sqrt{2}}$ (vii) $\frac{\pi-2}{4}$

(viii) $\frac{\pi}{4}-\frac{1}{2}\log 2$

10. (i) 14 (ii) $\frac{26}{3}$

(iii) 26 (v) $\frac{1}{3}\left(e^2-\frac{1}{e}\right)$

(iv) $\frac{1}{2}(127+e^8)$ (vi) 3

11. (i) $\frac{1}{5}\log\left|\frac{\tan x-2}{2\tan x+1}\right|+c$ (ii) $\frac{\pi}{8}\log 2$

(iii) $\frac{\pi}{2}\log\frac{1}{2}$

12. $\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log 2$

13. $\frac{-\pi}{2}\log 2$

14. $\log 2$

15. $\frac{1}{\sqrt{2}}\log\left|\sqrt{2}+1\right|$

SELF ASSESSMENT TEST-1

1. (c) $\frac{x^{17}}{17}-x+c$ 2. (d) $-\cos(x^2+2022)+c$ 3. (c) 5

4. (a) $\tan x+\cot x+c$ 5. (d) 4

SELF ASSESSMENT TEST-2

1. (a) 0 2. (c) π 3. (a) $x\log(x^2+1)-2x+2\tan^{-1}x+c$

4. (b) $\frac{e^x(\sin x-\cos x)}{2}+c$ 5. (c) 3

CHAPTER 8

APPLICATIONS OF INTEGRALS

In real life, integrations are used in various fields such as engineering, where engineers use integrals to find the shape of building. In Physics, used in the centre of gravity etc. In the field of graphical representation. Where three-dimensional models are demonstrated.

The PETRONAS TOWERS in KUALA LUMPUR experience high forces due to wind. Integration was used to create this design of building.

APPLICATIONS OF INTEGRALS

Topics to be covered as per C.B.S.E. revised syllabus (2025-26)

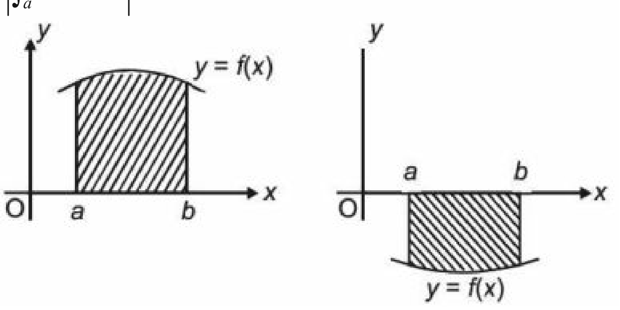
- Applications in finding the area under simple curves, especially lines, circles/ parabolas/ellipse (in standard form only)

POINTS TO REMEMBER

AREAS OF BOUNDED REGIONS

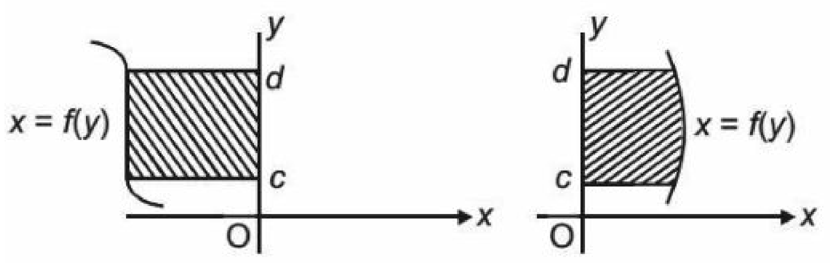
- Area bounded by the curve $y = f(x)$, the x axis and between the ordinates, $x = a$ and $x = b$ is given by

$$\left|\int_a^b f(x)dx\right|$$

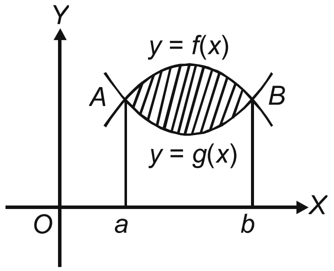


- Area bounded by the curve $x = f(y)$, the y-axis and between the abscissas, y = c and y = d is given by

$$\left|\int_c^d f(y)dy\right|$$



- Area bounded by two curves $y = f(x)$ and $y = g(x)$ such that $0 \le g(x) \le f(x)$ for all $x \in [a, b]$ and between the ordinates $x = a$ and $x = b$ is given by



$$\int_a^b [f(x) - g(x)]dx$$

- Area of the following shaded region = $\left|\int_a^k f(x)dx\right| + \int_k^b f(x)dx$

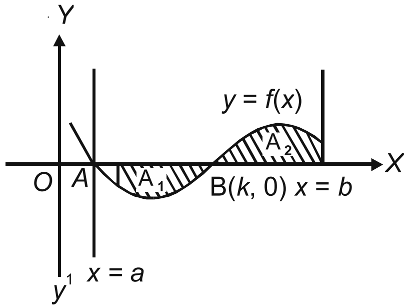
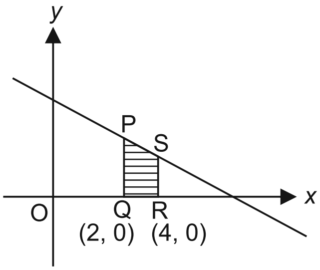


Illustration:

Using integration. Find the area of the region bounded by the line $2y + x = 8$, the x-axis and the lines $x = 2$ and $x = 4$

Solution: Required area= Areaof PQRS

= Area bounded by the line $2y + x = 8$, x-axis and ordinates $x = 2$, $x = 4$



$$= \int_2^4 y\,dx = \int_2^4 \frac{8-x}{2}\,dx$$

$$= \frac{1}{2}\left[8x - \frac{x^2}{2}\right]_2^4 = \frac{1}{2}[(32-8)-(16-2)]$$

$$= \frac{1}{2}[24-14] = \frac{1}{2}\times 10 = 5 \text{ sq.units}$$

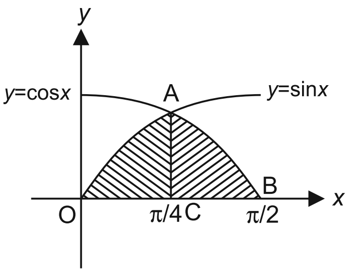
Illustration:

Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x-varies from 0 to $\pi/2$. Find the area of the region enclosed by the curves and the x-axis.

Solution: Given curves $y = \sin x$

and $y = \cos x$

Area of shaded region



$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$$

$$= -[\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2} = -\left[\frac{1}{\sqrt{2}} - 1\right] + \left[1 - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{-1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}} = (2-\sqrt{2}) \text{ square units}$$

Illustration:

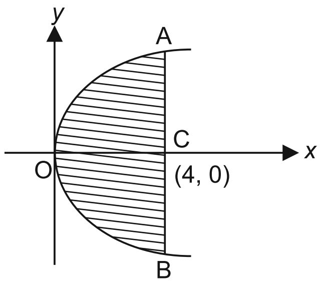
Using integration, find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.

Solution: Given curve $y^2 = 16x$

line $x = 4$

Area of shaded region

= 2(area of AOC)



$$= 2\int_0^4 y\,dx = 2\int_0^4 4\sqrt{x}\,dx$$

$$= 8 \times \frac{2}{3}\left[x^{3/2}\right]_0^4 = \frac{16}{3}[8] = \frac{128}{3} \text{ sq.units}$$

Illustration:

Using integration, find the area of the smaller portion of the circle $x^2 + y^2 = 4$ cut off by the line $x = 1$.

Solution: Circle $x^2 + y^2 = 4$

line $x = 1$

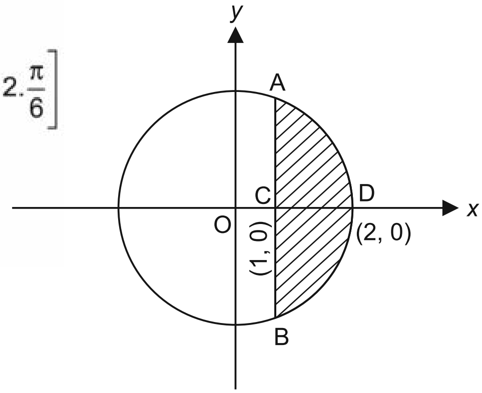
Area of shaded region

= 2(area bounded by the circle, the x-axis and ordinate $x = 1$ and $x = 2$)

$$= 2\int_1^2 y\,dx = 2\int_1^2 \sqrt{4-x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2 = 2\left[2.\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2.\frac{\pi}{6}\right]$$

$$= 2\left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right] = \frac{4\pi}{3} - \sqrt{3} \text{ sq. units}$$



ONE MARK QUESTIONS

Multiple Choice Questions (1 Mark Each)

Select the correct option out of the four given options:

1. The area of the region bounded by the curve $y = x^2$, x-axis and the lines $x = -1$, $x = 1$ is

(a) $\frac{1}{3}sq.$ units (b) $\frac{2}{3}sq.$ units

(c) 1 sq. unit (d) 2 sq. units

2. The area bounded by y = sin 2x, $0 \le x \le \frac{\pi}{4}$ and coordinate axes is

(a) $\frac{1}{2}$ sq. units (b) 1 sq. unit

(c) $\frac{3}{2}$ sq. units (d) 2 sq. units

3. The area bounded by the line x + 2y = 8 and the lines x = 1 and x = 3 is

(a) 16 sq. units (b) 8 sq. units

(c) 12 sq. units (d) 6 sq. units

4. The area enclosed by the parabola $y^2 = 8x$ and its latus rectum is

(a) $\frac{16}{3}$ sq. units (b) $\frac{64}{3}$ sq. units

(c) $\frac{32}{3}$ sq. units (d) $\frac{16\sqrt{2}}{3}$ sq. units

5. The area bounded by the curve y = cos x and x-axis between x = 0 and x = π is

(a) 0 sq. units (b) 1 sq. units

(c) 2 sq. units (d) 4 sq. units

ASSERTION-REASON BASED QUESTIONS

In the following qustions a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices:

(a) Both (A) and (R) are true and (R) is the correct explanation of (A)

(b) Both (A) and (R) are true, but (R) is not the correct explanation of (A)

(c) (A) is true and (R) is false

(d) (A) is false, but (R) is true

6. Assertion (A) : Area enclosed by the curve $x^2 + y^2 = 4$ is given by $4\int_0^2 \sqrt{4-x^2}\,dx$

Reason (R) : The curve $x^2 + y^2 = 4$ is symmetric about both the axes.

7. Assertion (A) : Area of the region bounded by the parabola $y^2 = 4x$ and its latus rectum is given by $2\int_0^1 2\sqrt{x}\,dx$

Reason (R) : Length of the latus rectum of the parabola $y^2 = 4ax$ is $4a$.

Using Integration:

TWO MARKS QUESTIONS

1. Find the area of the circle $x^2 + y^2 = 16$.
2. Find the area of the parabols $y^2 = 4a\,x$ bounded by its latus rectum.
3. Find the area bounded by the curve $y^2 = x$, x – axis and the lines $x = 0$, $x = 4$.
4. Find the area bounded by the region $\{(x, y): x^2 \le y \le |x|\}$.
5. Find the area bounded by the region $y = 9x^2$, $y = 1$ and $y = 4$.
6. Find the area bounded by the curve $y = \sin x$ between $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$
7. Find the area bounded by the lines $y = 2x + 3$, $y = 0$, $x = 2$ and $x = 4$.
8. Find the area of the region bounded by $y^2 = 4x$, $x = 1$, $x = 4$ and x-axis in the first quadrant.
9. Find the area bounded by the curves $y^2 = 4ax$ and the lines $y = 2a$ and y-axis.
10. Find the area of the triangle formed by the straight lines $y = 2x$, $x = 0$ and $y = 2$

THREE/FIVE MARKS QUESTIONS

Using Integration

1. Find the area bounded by the curve $4y = 3x^2$ and the line $3x - 2y + 12 = 0$.
2. Find the area bounded by the curve $x = y^2$ and the line $x + y = 2$.
3. Find the area of the triangular region whose vertices are (1, 2), (2 – 2) and (4, 3).
4. Find the area bounded by the region $\{(x, y): x^2 + y^2 \le 1 \le x + \frac{y}{2}\}$
5. Find the area of the region bounded by the lines $x - 2y = 1$, $3x - y - 3 = 0$ and $2x + y - 12 = 0$.
6. Prove that the curve $y = x^2$ and, $x = y^2$ divide the square bounded by $x = 0$, $y = 0$, $x = 1$, $y = 1$ into three equal parts.
7. Find the area of the smaller region enclosed between ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the line $bx + ay = ab$.
8. Using integration, find the area of the triangle whose sides are given by $2x + y = 4,\ 3x - 2y = 6\ and\ x - 3y + 5 = 0$.
9. Using integration, find the area of the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

10. Find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$.

11. Find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

12. Using integration, find the area of the region bounded by the line $x - y + 2 = 0$, the curve $x^2 = y$ and y-axis.

13. Using integration, find the area of the region bounded by the curve $y = 1 + |x + 1|$ and lines $x = -3, x = 3, y = 0$.

14. Find the area of the region enclosed between curves $y = |x - 1|$ and $y = 3 - |x|$.

15. If the area bounded by the parabola $y^2 = 16ax$ and the line $y = 4mx$ is $\frac{a^2}{12}$ sq unit then using integration find the value of m.

16. Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $y = x$ and x-axis in first quadrant.

17. Find the area bounded by the parabola $y^2 = 4x$ and the straight line $x + y = 3$.

18. Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

19. Find the area of region $\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \leq \frac{x}{3} + \frac{y}{2}\right\}$

20. Using integration, find the area of the triangle ABC, whose vertices are A(2, 5), B(4, 7) and C(6, 2).

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

   (a) $\frac{9}{2}$ sq. units (b) $\frac{9}{3}$ sq. units

   (c) $\frac{9}{4}$ sq. units (d) $\frac{9}{5}$ sq. units

2. Area lying in first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

   (a) π sq. units (b) $\frac{\pi}{3}$ sq. units

   (c) $\frac{\pi}{2}$ sq. units (d) $\frac{\pi}{4}$ sq. units

3. The area of the region bounded by the curve y = x + 1 and the lines x = 2, x = 3 and x-axis is

(a) $\frac{13}{2}$ sq.units (b) $\frac{11}{2}$ sq.units

(c) $\frac{9}{2}$ sq.units (d) $\frac{7}{2}$ sq.units

4. The area bounded by the curve $y^2 = x$ and the line $x = 2y$ is

(a) $\frac{1}{3}$ sq.units (b) $\frac{2}{3}$ sq.units

(c) 1 sq. unit (d) $\frac{4}{3}$ sq.units

5. The area of the region bounded by the $y = \sin x$, $y = \cos x$ and y-axis, $0 \le x \le \frac{\pi}{4}$ is

(a) $(\sqrt{2}+1)$ sq.units (b) $(\sqrt{2}-1)$ sq.units

(c) $2\sqrt{2}$ sq.units (d) $(2\sqrt{2}-1)$ sq.units

ANSWERS

ONE MARKS QUESTION

1. (b) $\frac{2}{3}$ square units
2. (a) $\frac{1}{2}$ square units.
3. (d) 6 square units
4. (c) $\frac{32}{3}$ square units.
5. (c) 2 square units
6. (a)
7. (b)

TWO MARKS QUESTIONS

1. 16π square units.
2. $\frac{8}{3}a^2$ square units.
3. $\frac{16}{3}$ square units.
4. $\frac{1}{3}$ square units.
5. $\frac{28}{9}$ square units.
6. 2 square units.
7. 18 square units.
8. $\frac{28}{3}$ square units.
9. $\frac{2}{3}a^2$ square units.
10. 1 square units.

THREE/FIVE MARKS QUESTIONS

1. 27 square units.
2. $\frac{9}{2}$ square units.
3. $\frac{13}{2}$ square units.

4. $\left(\frac{\pi}{4}-\frac{2}{5}-\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$ square units.

5. 10 square units.

7. $\left(\frac{\pi-2}{4}\right)ab$ square units.

8. 3.5 square units.

9. 4 square units.

10. $\left(\frac{\pi}{4}-\frac{1}{2}\right)$ square units.

11. $\frac{9}{8}$ square units.

12. $\frac{10}{3}$ square units.

13. 16 square units.

14. 4 square units.

15. m = 2.

16. 2π sq. units

17. $\frac{64}{3}$ sq. units

18. 9 sq. units

19. $\frac{3}{2}(\pi-2)$ sq. units

20. 7 sq. units

SELF ASSESSMENT TEST-1

1. (C) 2. (A) 3. (D) 4. (D) 5. (B)

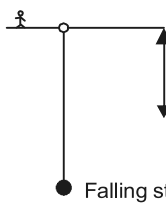
CHAPTER-9

DIFFERENTIAL EQUATIONS

Sky diving is a method of transiting from a high point in the atmosphere to the surface of the Earth with the aid of graity. This involves the control of speed during the descent using a parachute. Once the sky diver jumps from an airplane, the net force experienced by the diver can be calculated using

DIFFERENTIAL EQUATIONS.

Another eg.



D.E. is

$$my'' = mg$$

$$\Rightarrow \quad y'' = g = \text{constant}$$

where y = distance travelled by the stone at any time t.

and g = acceleration due to gravity.

TOPICS TO BE COVERED AS PER CBSE LATEST CURRICULUM 2025-26

- Definition, order and degree
- General and particular solutions of a D.E.
- Solutions of D.E. using method of separation of variables.
- Solutions of homogeneous differential equations of first order and first degree.
- Solutions of linear differential equations of the type.

$\frac{dy}{dx} + py = q,$ where p and q are functions of x or constants.

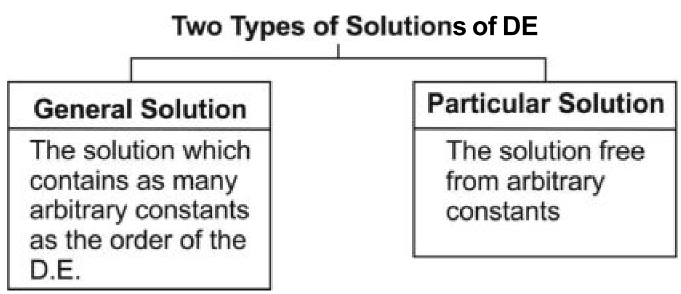
$\frac{dx}{dy} + px = q,$ where p and q are functions of y or constants.

KEY POINTS :

- **DIFFERENTIAL EQUATION :** It is an equation involving derivatives of the dependent variable w.r.t independent variables and the variables themselves.
- **ORDINARY DIFFERENTIAL EQUATION (O.D.E.):** A D.E. involving derivatives of the dependent variable w.r.t only one independent variable is an ordinary D.E.

  In class XII O.D.E. is referred to as D.E.

- **PARTIAL DIFFERENTIAL EQUATION (P.D.E.):** A Differential equation involving derivatives w.r.t more than one independent variables is called a partial D.E.
- **ORDER of a D.E. :** It is the order of the highest order derivative occurring in the D.E.
- **DEGREE of a D.E. :** It is the highest power of the highest order derivative occurring in the D.E. provided D.E. is a polynomial equation in its derivatives. It is always a Natural no.
- **SOLUTION OF THE D.E. :** A relation between involved variables, which satisfies the given D.E. is called its solution.



- **FORMATION OF A DIFFERENTIAL EQUATION :** We differentiate the function successively as many times as the number of arbitary constants in the given function and then eliminate the arbitiary constants from these equations.
- **ORDER of a D.E. :** It is equal to the number of arbitrary constants in the general solution of a D.E.

Solution of a First Order First Degree D.E.

- Variable separable method
- Homogeneous Differential equation
- Linear Differential equation

- **"VARIABLE SEPARABLE METHOD" :** It is used to solve D.E. in which variables can be separated completely i.e, terms containing x should remain with dx and terms containing y should remain with dy.
- **"HOMOGENEOUS DIFFERENTIAL EQUATION :** D.E. of the form $\frac{dy}{dx} = F(x, y)$ where *F(x, y)* is a homogeneous function of degree 0

i.e. $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$

or $F(\lambda x, \lambda y) = F(x, y)$ for some non-zero constant λ.

To solve this type put $y = vx$

To Solve homogenous D.E. of the type $\frac{dx}{dy} = G(x, y),$ we make substitution $x = vy$

- **LINEAR DIFFERENTIAL EQUATION :** P.D.E of the form $\frac{dy}{dx} + Py = Q$ where P and Q are constants or functions of x only is known as first order linear differential equation in y.

  Its solution is given as

  $y.(I.F.) = \int Q \times (I.F.)dx + C,$ where

  I. F = Integrating factor = $e^{\int P dx}$

  Another form of Linear Differential Equation is $\frac{dx}{dy} + P_1 x = Q_1$, where P_1 and Q_1 are constants or functions of y only, is known as first order linear differential equation in *x*.

  Its solution is given as

  $x.(I.F) = \int Q_1 X(I.F.)\, dy + C,$ where $I.F. = e^{\int P_1 dy}$

Illustration:

Write the order and degree of the Differential Equation

$$\left[1 + (y')^2\right]^{3/2} = ky''$$

Solution: Squaring both the sides

$$\left[1 + (y')^2\right]^3 = k^2 (y'')^2$$

$\therefore$ Order of D.E. = 2

and Degree of D.E. = 2

Illustration:

Solve the differential equations

$(1+e^{2x})dy + e^{x}(1+y^{2})dx = 0;\ y(0) = 1$

Solution: $\frac{dy}{dx} = \frac{-e^{x}(1+y^{2})}{1+e^{2x}}$

Using Variables separables method,

$$\frac{dy}{1+y^{2}} = \frac{-e^{x}}{1+e^{2x}}dx$$

Integrating both sides we get

$$\int \frac{1}{1+y^{2}}dy = -\int \frac{e^{x}}{1+e^{2x}}dx$$

$\Rightarrow \tan^{-1}y = -\int \frac{dt}{1+t^{2}}$; On putting $e^{x} = t$

$= -\tan^{-1}t$

$\Rightarrow \tan^{-1}y = -\tan^{-1}(e^{x}) + C$

$\Rightarrow \tan^{-1}y + \tan^{-1}(e^{x}) = C$

At $x = 0$, $y = 1$ given

$\therefore \tan^{-1}(1) + \tan^{-1}(1) = C$

$\Rightarrow \frac{\pi}{4}\times 2 = C$

$\Rightarrow C = \frac{\pi}{2}$

$\therefore$ Particular solution of D.E. is given by $\tan^{-1}y + \tan^{-1}(e^{x}) = \frac{\pi}{2}$.

Illustration:

Solve $(x-y)\frac{dy}{dx} = x + 2y$

Solution: $\frac{dy}{dx} = \frac{x+2y}{x-y} = f(x,y)$

Now $f(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \frac{\lambda(x+2y)}{\lambda(x-y)} = \lambda^{o}f(x, y)$

Clearly, f is homogeneous function in x and y.

So, given D.E. is **homogenous D.E.**

Now, Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + \frac{x\,dv}{dx}$$

$$\therefore \quad v + \frac{x\,dv}{dx} = \frac{x+2vx}{x-vx}$$

$$\Rightarrow v + \frac{x\,dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow \frac{x\,dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$\Rightarrow \frac{x\,dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \frac{(1-v)dv}{1+v+v^2} = \frac{dx}{x}$$

Integrating both sides we get

$$\Rightarrow -\frac{1}{2}\int \frac{2v-2+1-1}{1+v+v^2}dv = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\int \frac{2v+1}{1+v+v^2}dv + \frac{3}{2}\int \frac{1}{1+v+v^2}dv = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log|1+v+v^2| + \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dv = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log\left|1+\frac{y}{x}+\frac{y^2}{x^2}\right| + \left(\frac{3}{2}\right).\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log|x^2+xy+y^2| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = C$$

Illustration:

Find the particular solution of the differential equation

$\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y\,(y \neq 0)$ given that $x = 0$ when $y = \pi/2$.

Solution: Clearly, it is a Linear D.E. in x

$\frac{dx}{dy} + Px = Q$ where

$P = \cot y$, $Q = 2y + y^2 \cot y$

I.F. = $e^{\int Pdy}$ = $e^{\int \cot y\, dy}$ = $e^{\log(\sin y)}$ = sin y

$\therefore$ solution of D.E. is given by

x. (I.F) = $\int Q.\text{IF} dy$ + C; C is arbitrary constant

$\Rightarrow$ $x.(\sin y) = \int (2y + y^2 \cot y)\sin y\, dy + C$

$= \int 2y \sin y\, dy + \int y^2 \cos y\, dy + C$

$= \int 2y \sin y\, dy + y^2 .\sin y - \int 2y \sin y\, dy + C$

$\Rightarrow$ $x \sin y = y^2 \sin y + C$

Now, $x = 0$, when $y = \frac{\pi}{2}$

So, $0 = \frac{\pi^2}{4} + C \Rightarrow C = -\frac{\pi^2}{4}$

$\therefore$ $x \sin y = y^2 \sin y - \frac{\pi^2}{4}$

or $\boxed{x = y^2 - \frac{\pi^2}{4} \operatorname{cosec} y}$

ONE MARK QUESTIONS

1. The general solution of the D.E.

   $y\,dx - xdy = 0$; (Given x, $y > 0$), is of the form.

   (a) $xy = c$ (b) $x = cy^2$

   (c) $y = cx$ (d) $y = cx^2$

   (Where 'c' is an orbitary positive constant of integration)

2. The differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$ determines a family of circles with

   (a) Variable radii and fixed centre (0, 1)

   (b) Variable radii and fixed centre (0, –1)

   (c) Fixed radius 1 and variable centre on x-axis

   (d) Fixed radius 1 and variable centre on y-axis

3. The solution of the D.E. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ is

   (a) $e^x = \frac{y^3}{3} + e^y + c$ (b) $e^y = \frac{x^2}{3} + e^x + c$

   (c) $e^y = \frac{x^3}{3} + e^x + c$ (d) None f these

4. The order and degree of the D.E. $\frac{d^4y}{dx^4} + \sin(y''') = 0$ are respectively

   (a) 4 and 1 (b) 1 and 2

   (c) 4 and 4 (d) 4 and not defined

5. A homogeneous differential equation of the type $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution.

(a) y = vx (b) v = yx

(c) x = vy (d) x = v

6. Integrating factor of the D.E. $\frac{dy}{dx} + y\tan x - \sec x = 0$ is

(a) cosx (b) secx

(c) $e^{\cos x}$ (d) $e^{\sec x}$

7. The order and degree of the D.E. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{4}} + x^{\frac{1}{6}} = 0,$ respectively are

(a) 2 and 4 (b) 2 and 2

(c) 2 and 3 (c) 3 and 3

8. The order of the D.E. of a family of curves resprsented by an equation containing four arbitrary constants, will be

(a) 2 (b) 4

(c) 6 (d) None of these

9. An equation which involves variable as well as derivatives of the dependent variable w.r.t. the independent variable, is known as

(a) differential equation (b) integral equation

(c) linear equation (d) quadantic equation

10. $\tan^{-1}x + \tan^{-1}y = c$ is general solution of the D.E.

(a) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ (b) $\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$

(c) $(1+x^2)dy + (1+y^2)dx = 0$ (d) $(1+x^2)dx + (1+y^2)dy = 0$

11. The particular solution of $\log\left(\frac{dy}{dx}\right) = 3x + 4y,\ y(0) = 0$ is

(a) $e^{3x} + 3e^{-4y} = 4$ (b) $4e^{4x} + 3e^{-4y} = 3$

(c) $3e^{3x} - 4e^{4y} = 7$ (d) $4e^{3x} + 3e^{-4y} = 7$

12. The solution of the equation $\frac{dy}{dx} = \frac{3x-4y-2}{3x-4y-3}$ is

(a) $(x-y^2)+c = \log(3x-4y+1)$ (b) $x-y+c = \log(3x-4y+4)$

(c) $(x-y+c) = \log(3x-4y-3)$ (d) $x-y+c = \log(3x-4y+1)$

13. If $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

(a) $y\log\left(\dfrac{x}{y}\right) = cx$

(b) $x\log\left(\dfrac{y}{x}\right) = cy$

(c) $\log\left(\dfrac{y}{x}\right) = cx$

(d) $\log\left(\dfrac{x}{y}\right) = cy$

14. Solution of D.E. xdy – ydx = 0 respresents

(a) rectangular hyperbola

(b) parabola whose vertex is at orgain

(c) circle whose centre is at origin

(d) straight line passing through origin

15. Family $y = bx + c^4$ of curves will correspond to a differential equation of order

(a) 3

(b) 2

(c) 1

(d) infinite

16. The integrating factor of the differential equation $(1-y^2)\dfrac{dx}{dy} + yx = ay, (-1 < y < 1)$ is :

(a) $\dfrac{1}{y^2-1}$

(b) $\dfrac{1}{\sqrt{y^2-1}}$

(c) $\dfrac{1}{1-y^2}$

(d) $\dfrac{1}{\sqrt{1-y^2}}$

17. The general solution of the differential equation $xdy - (1+x^2)dx = 0$ is

(a) $y = 2x + \dfrac{x^3}{3} + c$

(b) $y = 2\log x + \dfrac{x^3}{3} + c$

(c) $y = \dfrac{x^2}{2} + c$

(d) $y = \log x + \dfrac{x^2}{2} + c$

ASSERTION REASON TYPE QUESTIONS

Directions : Each of these questions contains two statements, Assertion (A) and Reason (R) Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explantion of (A)

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A)

(c) (A) is true and (R) is false

(d) (A) is false but (R) is true

18. Assertion (A) : Order of the differential equation whose solution is $y = c_1 e^{x+c_2} + c_3 e^{x+c_4}$ is 4.

    Reason (R) : Order of the differential equation is equal to the number of independent arbitrary constant mentioned in the solution of differential equation.

19. Assertion (A) : The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$ is not defined.

    Reason (R) : If the differential equation is a polynomial in terms of its derivatives, then its degree is defined.

20. Assertion (A) : $\frac{dy}{dx} = \frac{x^3 - xy^2 + y^3}{x^2y - x^3}$ is a homogeneous differential equation.

    Reason (R) : The function $f(x, y) = \frac{x^3 - xy^2 + y^3}{x^2y - x^3}$ is homogeneous.

TWO MARKS QUESTIONS

1. Write the general solution of the following D.Eqns.

(i) $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$ (ii) $\frac{dy}{dx} = \frac{1 - cos\,2x}{1 + cos\,2y}$

(iii) $(e^x + e^{-x})\,dy = (e^x - e^{-x})dx$

2. Given that $\frac{dy}{dx} = e^{-2y}$ and y = 0 when x = 5.

Find the value of x when y = 3.

3. Name the curve for which the slope of the tangent at any point is equal to the ratio of the abbcissa to the ordinate of the point.

4. Solve $\frac{xdy}{dx} + y = e^x$.

THREE MARKS QUESTIONS

1. (i) Show that $y = e^{m\sin^{-1}x}$ is a solution of $(1 - x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} - m^2y = 0$

(ii) Show that $y = a\,cos(log\,x) + b\,sin(log\,x)$ is a solution of

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} + y = 0$$

(iii) Verify that y = log $\left(x + \sqrt{x^2 + a^2}\right)$ satisfies the D.E.

$$\left(a^2 + x^2\right)y'' + xy' = 0$$

2. Solve the following differential equations.

(i) $xdy - (y + 2x^2)dx = 0$

(ii) $(1 + y^2)\,tan^{-1}x\;dx + 2y(1 + x^2)dy = 0$

(iii) $x^2\frac{dy}{dx} = x^2 + xy + y^2$

(iv) $\frac{dy}{dx} = 1 + x + y^2 + xy^2$, $y = 0$ when $x = 0$

(v) $xdy - ydx = \sqrt{x^2 + y^2}dx, y = 0$ when $x = 1$

3. Solve each of the following differential equations

(i) $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0,\ y(0) = 0$

(ii) $(x+1)\frac{dy}{dx} = 2e^{-y} - 1, y(0) = 0$

(iii) $e^x \tan y dx + (2 - e^x)\sec^2 y dy = 0,\ y\ (0) = \frac{\pi}{4}$

(iv) $(x^2 - y^2)dx + 2xydy = 0$

(v) $(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}, y = 0$ when $x = 1$

4. Solve the following differential equations

(i) Find the particular solution of

$2y\ e^{x/y}\ dx + (y - 2xe^{x/y})\ dy = 0, x = 0$ if $y = 1$

(ii) $x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$

(iii) $\frac{dy}{dx}$ = cos $(x + y)$ + sin $(x + y)$

[**Hint :** Put $x + y = z$]

(iv) Show that the Differential Equation $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is homogenous and also solve it.

(v) $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}, |\,x\,| \neq 1$

FIVE MARKS QUESTIONS

Q. 1 Solve $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

Q. 2 Solve $(x\ dy - ydx)y \sin\left(\frac{y}{x}\right) = (ydx + xdy)x \cos\left(\frac{y}{x}\right)$

Q. 3 Find the particular solution of the D.E. $(x-y)\frac{dy}{dx} = x+2y$ given that $y=0$ when $x=1$.

Q. 4 Solve $dy = \cos x\,(2 - y\operatorname{cosec} x)dx,$ given that $y=2$ when $x=\pi/2$

Q. 5 Find the particular solution of the D.E. $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$ given that $y=0$ when $x=1$

CASE STUDY QUESTIONS

1. An equation involving derivatives of the dependent variable w.r.t. the independent variables is called a differential equation. A differential equation of the from $\frac{dy}{dx} = f(x,y)$ is said to be homogeneous if f(x, y) is a homogeneous function of degree zero, whereas a function f(x, y) is a homogeneous function of degree *n* if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. To solve a homogeneous differential equation of the type $\frac{dy}{dx} = f(x,y) = g\left(\frac{y}{x}\right)$ we make the substitution $y = vx$ and then separate the variables.

Based on the above, answer the following quations:

(i) Show that $x^2 y\,dx - (x^3+y^3)\,dy = 0$ is a differential equation of the type

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

(ii) Solve the above equation to find its general solution.

Self Assessment Test-1 Differential Equations

Q. 1 The general solution of the D.E.

$$\log\left(\frac{dy}{dx}\right) = ax + by \text{ is}$$

(a) $\frac{-e^{-by}}{b} = \frac{e^{ax}}{a} + C$

(b) $e^{ax} - e^{-by} = C$

(c) $be^{ax} + ae^{by} = C$

(d) none of these

Q. 2 The general solution of the DE

$$x^2\frac{dy}{dx} = x^2 + xy + y^2 \text{ is}$$

(a) $\tan^{-1}\left(\frac{y}{x}\right) = \log x + c$

(b) $\tan^{-1}\left(\frac{x}{y}\right) = \log x + c$

(c) $\tan^{-1}\left(\frac{y}{x}\right) = \log y + c$

(d) none of these

Q. 3 The solution of the D.E.

$dy = (4 + y^2)\, dx$ is

(a) $y = 2\tan(x + c)$

(b) $y = 2\tan(2x + c)$

(c) $2y = \tan(2x + c)$

(d) $2y = 2\tan(x + c)$

Q. 4 What is the degree of the D.E.

$$y = x\left(\frac{dy}{dx}\right)^3 + \left(\frac{dy}{dx}\right)^2$$

(a) 1

(b) 3

(c) –2

(d) Degree doesn't exist

Q. 5 Solution of D.E. $xdy - ydx = 0$ represents:

(a) a rectangular hyperbola

(b) a parabola whose vertex is at the origin

(c) a straight line passing through the origin

(d) a circle whose centre is at the origin.

Self Assessment Test-2

Q. 1 The solution of the D.E. $x\frac{dy}{dx} + 2y = x^2$ is

(a) $y = \frac{x^2 + c}{4x^2}$ (b) $y = \frac{x^2}{4} + c$

(c) $y = \frac{x^2 + c}{x^2}$ (d) $y = \frac{x^4 + c}{4x^2}$

Q. 2 The solution of the $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$, is

(a) $y = e^{-x}(x-1)$ (b) $y = x e^{x}$

(c) $y = xe^{-x} + 1$ (d) $y = x e^{-x}$

Q. 3 If $\frac{dy}{dx} = \frac{2^{x+y} - 2^x}{2^y}$, $y(0) = 1$, then $y(1)$ is equal to **[JEE mains 2021]**

(a) $\log_2(2 + e)$ (b) $\log_2(1 + e)$

(c) $\log_2(2e)$ (d) $\log_2(1 + e^2)$

Q. 4 If the solution curve of the D.E. $(2x - 10y^3)\,dy + y\,dx = 0$ passess through the points (0, 1) and (2, β), then β is a root of the equation

(a) $y^5 - 2y - 2 = 0$ (b) $2y^5 - 2y - 1 = 0$

(c) $2y^5 - y^2 - 2 = 0$ (d) $y^5 - y^2 - 1 = 0$ **[JEE mains 2021]**

Q. 5 Consider a curve $y = f(x)$ passing through the point (–2, 2) and the slope of the tangent to the curve at any point $(x, f(x))$ is given by

$f(x) + \frac{x\,df(x)}{dx} = x^2$,

(a) $x^3 + 2x\,f(x) - 12 = 0$ (b) $x^3 + xf(x) + 12 = 0$

(c) $x^3 - 3x\,f(x) - 4 = 0$ (d) $x^2 + 2xf(x) + 4 = 0$ **(HOTS)**

Answers

ONE MARK QUESTIONS

1. (c) y = cx	2. (c)	3. (c)	4. (d)
5. (c) x = vy	6. (b)	7. (a)	8. (b) 4
9. (a)	10. (c)	11. (d)	12. (d)

13. (c) 14. (d) 15. (b) 2 16. (d) $\frac{1}{\sqrt{1-y^2}}$

17. (d) 18. (d) 19. (a) 20. (a)

TWO MARKS QUESTIONS

1. (i) $y = \frac{x^6}{6} + \frac{x^3}{3} - 2\,log\,|x| + C$ (ii) $2(y-x) + \sin 2y + \sin 2x = c$

   (iii) $y = \log_e |e^x + e^{-x}| + C$

2. $\frac{e^6+9}{2}$ 3. Rectangular hyperbola 4. $y.x = e^x + c$

THREE MARKS QUESTIONS

2. (i) $y = 2x^2 + cx$ (ii) $\frac{1}{2}(\tan^{-1}x)^2 + \log(1+y^2) = c$

   (iii) $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$ (iv) $y = tan\left(x + \frac{x^2}{2}\right)$

   (v) $y + \sqrt{x^2+y^2} = x^2$

3. (i) $\left(1+x^2\right)y = \frac{4x^3}{3}$ (ii) $(2-e^y)(x+1) = 1$

   (iii) $\tan y = 2 - e^x$ (iv) $x^2 + y^2 = cx$

   (v) $(1+x^2)y = tan^{-1}x - \pi/4$

4. (i) $e^{x/y} = \frac{-1}{2}\log|y| + 1$ (ii) $sin(y/x) = log\,|x| + c$

   (iii) $\log\left|1+\tan\left(\frac{x+y}{2}\right)\right| = x + c$ (iv) $\frac{y}{x} - log\,|y| = c$

   (v) $(x^2-1)y = \frac{1}{2}log\left|\frac{x-1}{x+1}\right| + c$

FIVE MARKS QUESTIONS

1. $y = -cos\,x + \frac{2\,sin\,x}{x} + \frac{2\,cos\,x}{x^2} + \frac{x\,log\,x}{3} - \frac{x}{9} + \frac{c}{x^2}$

2. $xy\,cos\left(\frac{y}{x}\right) = c$

3. $\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)-\frac{1}{2}\log|x^2+xy+y^2|=\frac{\sqrt{3}\pi}{6}$

4. $y\sin x=\frac{-1}{2}\cos(2x)+\frac{3}{2}$

5. $x=\frac{1}{2}e^{\tan^{-1}y}+\frac{1}{2}e^{-\tan^{-1}y}$

CASE STUDY QUESTIONS

1. (ii) $\frac{-x^3}{3y^3}$ + log |y| = c, c is an arbitrary constant

SELF ASSESSMENT TEST-1

1. (a)	2. (a)	3. (b)
4. (b)	5. (c)	

SELF ASSESSMENT TEST-2

1. (d)	2. (d)	3. (b)
4. (d)	5. (c)	

CHAPTER 10

VECTORS

Vectors are probably the most important tool to learn in all of physics and engineering. Vectors are used in daily life. Following are few of the examples.

- Navigating by air and by boat is generally done using vectors.
- Planes are given a vector to travel, and they use their speed to determine how far they need to go before turning or landing. Flight plans are made using a series of vectors.
- Sports instructions are based on using vectors.

VECTORS

Topics to be covered as per C.B.S.E. revised syllabus (2025-26)

- Vectors and scalars
- Magnitude and direction of a vector
- Direction consines and direction ratios of a vector.
- Types of vectors (equal, unit, zero, parallel and collinear vectors)
- Position vector of a point
- Negative of a vector
- Components of a vector
- Addition of vectors
- Multiplication of a vector by a scalar
- Position vector of a point dividing a line segment in a given ratio
- Definition, Geometrical interpretation, properties and application of scalar (dot) product of vectors
- Vector (cross) product of vectors.

POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a vector. It is denoted by a directed line segment.

- Two or more vectors which are parallel to same line are called collinear vectors.

- Position vector of a point P(a, b, c) w.r.t. origin (0, 0, 0) is denoted by $\overrightarrow{OP}$ where $\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$ and $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$.

- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space, then

  $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and

  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- Any vector $\vec{a}$ is called unit vector if $|\vec{a}| = 1$ It is denoted by $\hat{a}$

- If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two sides of a triangle in order, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by third side of a triangle taken in opposite order. This is called triangle law of addition of vectors.

- If $\vec{a}$ is any vector and λ is a scalar, then $\lambda\,\vec{a}$ is vector collinear with $\vec{a}$ and $|\lambda\,\vec{a}| = |\lambda||\vec{a}|$.

- If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then $\vec{a} = \lambda\,\vec{b}$ where λ is some non-zero scalar.

- Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\hat{a}$ where $\hat{a}$ is a unit vector in the direction of $\vec{a}$.

- If $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B, and C is any point which divides $\overrightarrow{AB}$ in ratio m:n internally then position vector $\vec{c}$ of point C is given as $\vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n}$. If C divides $\overrightarrow{AB}$ in ratio m:n externally, then $\vec{c} = \frac{m\vec{b}-n\vec{a}}{m-n}$. If C is mid point then $\vec{c} = \frac{\vec{a}+\vec{b}}{2}$

- The angles α, β *and* γ made by $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with positive direction of x, y and z-axis are called direction angles and cosines of these angles are called direction cosines of $\vec{r}$ usually denoted as $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

  Also $l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|}$ and $l^2 + m^2 + n^2 = 1$

  or $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

- The numbers a, b, c proportional to l, m, n are called direction ratios.
- Scalar product or dot product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a}\,.\,\vec{b}$ and is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$, θ is the angle between $\vec{a}$ *and* $\vec{b}$.$(0 \leq \theta \leq \pi)$.

- Dot product of two vectors is commutative i.e. $\vec{a}.\vec{b} = \vec{b}\,.\,\vec{a}$

- $\vec{a}.\vec{b} = 0 \iff \vec{a} = \vec{o}$ or $\vec{b} = \vec{o}$ or $\vec{a} \perp \vec{b}$.

- $\vec{a}\,.\,\vec{a} = |\vec{a}|^2$, so $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}\,.\,\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

- Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

  Projection vector of $\vec{a}$ along $\vec{b} = \left(\frac{(\vec{a}.\vec{b})}{|\vec{b}|}\right)\hat{b}$.

- Cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$. where θ is the angle between $\vec{a}$ and $\vec{b}$.($0 \leq \theta \leq \pi$). And $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}\,.\,\vec{b}$ and $\hat{n}$ form a right handed system.

- Cross product of two vectors is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, but $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

- $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} = \vec{0}, \vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$.

- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$.

- $\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}$ and $\hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath}, \hat{\imath} \times \hat{k} = -\hat{\jmath}$

- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \pm\left(\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}\right)$.

- $|\vec{a} \times \vec{b}|$ is the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

- $\frac{1}{2}|\vec{a} \times \vec{b}|$ is the area of parallelogram where diagonals are $\vec{a}$ and $\vec{b}$.

- If $\vec{a}, \vec{b}$ and $\vec{c}$ form a triangle, then area of the triangle

  $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{2}|\vec{c} \times \vec{a}|$.

Illustration:

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$ Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 27$

Solution:

$\because$ $\vec{d}$ is perpendicular to $\vec{a}$ and $\vec{b}$ *both*

$$Let\ \vec{d} = \lambda(\vec{a} \times \vec{b}) = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$\vec{d} = \lambda(32\hat{i} - \hat{j} - 14\hat{k})$$

But $\vec{c} \cdot \vec{d} = 27$

$\therefore$ $(2\hat{i} - \hat{j} + 4\hat{k}) . \lambda(32\hat{i} - \hat{j} - 14\hat{k}) = 27$

$\Rightarrow$ $\lambda(64 + 1 - 56) = 27$

$\Rightarrow$ $\lambda = 3$

and $\vec{d} = 3(32\hat{i} - \hat{j} - 14\hat{k}) = 96\hat{i} - 3\hat{j} - 42\hat{k}$

Illustration:

Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 5, |\vec{b}| = 7$ and $|\vec{c}| = 3$. Find the angle between $\vec{a}$ and $\vec{c}$

Solution:

Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\vec{a} + \vec{c} = -\vec{b}$$

$$(\vec{a} + \vec{c}) \cdot (\vec{a} + \vec{c}) = (-\vec{b}) \cdot -(\vec{b})$$

$\Rightarrow |\vec{a}|^2 + \vec{a} \cdot \vec{c} + \vec{c} \cdot \vec{a} + |\vec{c}|^2 = |\vec{b}|^2$ $\quad (\because \vec{a} \cdot \vec{a} = |\vec{a}|^2)$

$\Rightarrow$ $2\vec{a} \cdot \vec{c} = |\vec{b}|^2 - |\vec{a}|^2 - |\vec{c}|^2$

$\Rightarrow$ $2|\vec{a}||\vec{c}|\cos\theta = |\vec{b}|^2 - |\vec{a}|^2 - |\vec{c}|^2$

Where 'θ' be the angle between $\vec{a}$ and $\vec{c}$

$\Rightarrow 2 \times 5 \times 3 \cos\theta = 49 - 25 - 9$

$\Rightarrow \cos\theta = \frac{15}{30}$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

Illustration:

Let $\vec{a}$ and $\vec{b}$ are two unit vectors and 'θ' is the angle between them, then find 'θ' if $\vec{a}+\vec{b}$ is unit vector.

Solution:

Here $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = 1$

$\because |\vec{a}+\vec{b}|^2 = 1$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = 1$ $\qquad (\because \vec{a}\cdot\vec{a} = |\vec{a}|^2)$

$\Rightarrow |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + |\vec{b}|^2 = 1$

$\Rightarrow 2|\vec{a}||\vec{b}|\cos\theta = -1$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}.$

ONE MARK QUESTIONS

MULTIPLE CHOICE QUESTIONS (1 Mark Each)

Select the correct option out of the four given options:

1. If $\overrightarrow{AB} = 3\hat{i}+2\hat{j}-\hat{k}$ and the coordinate of A are (4, 1, 1), then the coordinate of B are.

   (a) (1, – 1, 2) (b) (–7, –3, 0)

   (c) (7, 3, 0) (d) (–1, 1, –2)

2. Let $\vec{a} = -2\hat{i}+\hat{j}, \vec{b} = \hat{i}+2\hat{j}$ and $\vec{c} = 4\hat{i}+3\hat{j}$, then the values of x and y such that $\vec{c} = x\vec{a}+y\vec{b}$, are:

   (a) x = 1, y = 2 (b) x = –1, y = 2

   (c) x = –1, y = –2 (d) x = 1, y = –1

3. A unit vector in the direction of the resultant of the vector $\hat{i}-\hat{j}+3\hat{k}, 2\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}+2\hat{j}-2\hat{k}$ is

   (a) $\frac{1}{\sqrt{21}}(4\hat{i}-2\hat{j}-\hat{k})$ (b) $\frac{1}{\sqrt{21}}(4\hat{i}-2\hat{j}+\hat{k})$

   (c) $4\hat{i}-2\hat{j}-\hat{k}$ (d) $\frac{1}{\sqrt{21}}(4\hat{i}+2\hat{j}-\hat{k})$

4. If $2\hat{i}+3\hat{j}+\hat{k}$ and $\hat{i}-2\hat{j}-\hat{k}$ are two vectors, then a vector of magnitude 5 units parallel to the sum of given vectors

(a) $\sqrt{\frac{5}{2}}(3\hat{i}+\hat{j})$ (b) $\frac{1}{\sqrt{30}}(\hat{i}+5\hat{j}+2\hat{k})$

(c) $\frac{1}{\sqrt{10}}(3\hat{i}+\hat{j})$ (d) $5(3\hat{i}+\hat{j})$

5. If $\vec{a} = \lambda\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b} = 5\hat{i}-9\hat{j}+2\hat{k}$ are perpendicular, then the value of 'λ' is:

(a) $\lambda = \frac{16}{5}$ (b) $\lambda = -\frac{16}{5}$

(c) $\lambda = 4$ (d) $\lambda = \frac{10}{9}$

6. The value of p for which $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}+p\hat{j}+3\hat{k}$ are parallel vector is

(a) $p = -\frac{30}{2}$ (b) p = 15

(c) $p = \frac{2}{3}$ (d) $p = \frac{3}{2}$

7. If $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}-3\hat{j}+p\hat{k}) = \vec{0}$, then the value of 'p' is

(a) $p = -\frac{20}{27}$ (b) $p = \frac{27}{2}$

(c) p = 0 (d) $p = \frac{27}{2}$

8. Value of $\hat{i}.(\hat{j}\times\hat{k})-(\hat{i}\times\hat{k}).\hat{j}$ is

(a) 2 (b) 1

(c) 0 (d) –2

9. If $\bar{a} = 5\hat{i}-4\hat{j}+\hat{k}, \bar{b} = -4\hat{i}+3\hat{j}-2\hat{k}$ and $\bar{c} = \hat{i}-2\hat{j}-2\hat{k}$ than the value of $\bar{c}.(\bar{a}\times\bar{b})$ is

(a) –5 (b) 5

(c) 35 (d) 30

10. If vector $\lambda\hat{i} + 3\hat{j}$ and $4\hat{i} + \lambda\hat{j}$ are collinear, then the value of 'λ' is

(a) $\lambda = 0$ (b) $\lambda = 4$

(c) $\lambda = 3$ (d) $\lambda = \pm 2\sqrt{3}$

11. A unit vector perpendicular to $2\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} - \hat{j} + 3\hat{k}$ is

(a) $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$ (b) $\frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$

(c) $\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$ (d) $\frac{1}{3}(\hat{i} + 2\hat{j} - 3\hat{k})$

12. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) 30° (b) 45°

(c) 60° (d) 90°

13. If $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$ are the diagonals of a parallelagram, then the area of the parallelogram is

(a) 8 sq. units (b) $\sqrt{91}$ sq. units

(c) $5\sqrt{3}$ sq. units (d) $10\sqrt{3}$ sq. units

14. If scalar projection of $\lambda\hat{i} + \hat{j} + 4\hat{k}$ on $2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units, then the value of λ is

(a) $\lambda = 5$ (b) $\lambda = -5$

(c) $\lambda = -9$ (d) $\lambda = 9$

15. If $\vec{a} . \vec{b} = 3$ and $|\vec{a} \times \vec{b}| = 3\sqrt{3}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) 30° (b) 60°

(c) 120° (d) 45°

16. If $|\vec{a}| = 4$ and $-3 \leq k \leq 2$, than the range of $|k\vec{a}|$ is

(a) [0, 12] (b) [8, 12]

(c) [0, 8] (d) [–12, 8]

17. If $|\vec{a}| = 4, |\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 10$, than the value of $|\vec{a} . \vec{b}|^2$ is

(a) 22 (b) 44

(c) 88 (d) None of these

18. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3, |\vec{b}| = 4$ and $|\vec{c}| = \sqrt{37}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

19. If $(\vec{a}+\vec{b}) \perp \vec{b}$ and $(\vec{a}+2\vec{b}) \perp \vec{a}$, then

(a) $(\vec{a}) = 2|\vec{b}|$
(b) $2|\vec{a}| = \vec{b}$
(c) $(\vec{a}) = (\vec{b})$
(d) $|\vec{a}| = \sqrt{2}|\vec{b}|$

20. If $|\vec{a}| = |\vec{b}| = |\vec{a}+\vec{b}| = 1$, then the value of $|\vec{a}-\vec{b}|$ is

(a) 0
(b) 1
(c) $\sqrt{3}$
(d) 2

Assertion-Reason Based Questions

In the following questions a statement of Assertion (A) is followed by a statement of Reason (R) Choose the correct answer out of the following couces:

(a) Both (A) and (R) are true and (R) is the correct explanation of (A)
(b) Both (A) and (R) are true, but (R) is not the correct explanation of (A)
(c) (A) is true and (R) is false
(d) (A) is false, but (R) is true

21. Assertion (A) : If $|\vec{a}| = 3, |\vec{b}| = 5$ and $\vec{a}\,.\,\vec{b} = 10$,

$$|\vec{a}\times\vec{b}|^2 = 125$$

Reason (R) : $|\vec{a}\times\vec{b}|^2 - (\vec{a}\,.\,\vec{b})^2 = ||\vec{a}|^2|\vec{b}|^{2\ 2}$

22. Assertion (A) : If $\vec{a}$ and $\vec{b}$ are unit vector such that $|\vec{a}+\vec{b}| = \sqrt{3}$, then the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$

Reason (R) : Angle between vectors $\vec{a}$ and $\vec{b}$ is given by cos θ = $\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$

23. Assertion (A) : If $|\vec{a}| = 4, |\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 20$, then $\vec{a} \perp \vec{b}$

Reason (R) : Two non zero vector $\vec{a}$ and $\vec{b}$ are perpandicular if $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|$

24. Assertion (A) : If $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 12$ and $\vec{a} = 2|\vec{b}|$, then $|\vec{a}| = 4$ and $|\vec{b}| = 2$

Reason (R) : If $\vec{a}$ and $\vec{b}$ are two vectors, then $(\vec{a}+\vec{b})\,.\,(\vec{a}-\vec{b}) = |\vec{a}|^2 - |b|^2$

25. Assertion (A) : If $|2\vec{a}+\vec{b}| = |2\vec{a}-\vec{b}|$, than $\vec{a}$ parellel to $\vec{b}$

Reason (B) : Two non zero vector $\vec{a}$ and $\vec{b}$ are perpendicular if $\vec{a}\,.\,\vec{b} = 0$.

TWO MARK QUESTIONS

1. A vector $\vec{r}$ is inclined to x – axis at 45° and y-axis at 60° if | $\vec{r}$ | = 8 units. find $\vec{r}$.

2. if $|\vec{a}+\vec{b}|$ = 60, $|\vec{a}-\vec{b}|$ = 40 and $|\vec{b}|$ = 46 find | $\vec{a}$ |

3. Write the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where

   $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

4. If the points (–1, –1, 2), (2, m, 5) and (3, 11, 6) are collinear, find the value of *m*.

5. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ write value of the following.
   $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

6. If $(\vec{a} \times \vec{b})^2 + (\vec{a}.\vec{b})^2$ = 144 and | $\vec{a}$ | = 4. Find the value of | $\vec{b}$ |.

7. If for any two vectors $\vec{a}$ and $\vec{b}$,

   $(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = \lambda\left[(\vec{a})^2 + (\vec{b})^2\right]$ then write the value of λ .

8. if $\vec{a}, \vec{b}$ are two vectors such that $|(\vec{a}+\vec{b})|$ = | $\vec{a}$ | then prove that 2 $\vec{a}$ + $\vec{b}$ is perpendicular to $\vec{b}$.

9. Show that vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

   $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}, \vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angle triangle.

10. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}$ = 0 and | $\vec{a}$ | = 5, | $\vec{b}$ | = 12, | $\vec{c}$ | = 13, then find $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

11. The two vectors $\hat{i} + \hat{j}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC respectively of Δ ABC, find the length of median through A.

12. If position vectors of the points A, B and C are $\vec{a}, \vec{b}$ and $4\vec{a} - 3\vec{b}$ respectively, then find vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

13. If position vectors of three points A, B and C are $-2\vec{a} + 3\vec{b} + 5\vec{c}, \vec{a} + 2\vec{b} + 3\vec{c}$ and $7\vec{a} - \vec{c}$ respectively. Then prove that A, B and C are collinear.

14. If the vector $\hat{i} + p\hat{j} + 3\hat{k}$ is rotated through an angle θ and is doubled in magnitude, then it becomes $4\hat{i} + (4p - 2)\hat{j} + 2\hat{k}$. Find the value of p.

15. If $\overrightarrow{AB} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ and $\overrightarrow{AC} = 3\hat{i} + 4\hat{k}$ are sides of the triangle ABC. Find the length of median through A.

16. Find scalar projection of the vector $7\hat{i} + \hat{j} + 4\hat{k}$ on the vector $2\hat{i} + 6\hat{j} + 3\hat{k}$. Also find vector porojection

17. Let $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular and $|\vec{a}| = |\vec{b}|$. Find x and y.

18. If $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a}$ and $\vec{b}$ so that $\vec{a} - \sqrt{2}\, b$ is a unit vector.

19. If $\vec{a} = 2\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$. Find the angle between $\vec{a}$ and $\vec{a} \times \vec{b}$.

20. Using vectors, prove that angle in a semi circle is 90°.

THREE MARKS QUESTIONS

1. The points A,B and C with position vectors $3\hat{\imath} - y\hat{\jmath} + 2\hat{k},\ 5\hat{\imath} - \hat{\jmath} + \hat{k}\ and\ 3x\hat{\imath} + 3\hat{\jmath} - \hat{k}$ are collinear. Find the values of x and y and also the ratio in which the point B divides AC.

2. If sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

3. Let $\vec{a} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k},\ \vec{b} = \hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}\ and\ \vec{b}$ and satisfying $\vec{d}.\vec{c} = 21$

4. If $\hat{a}\ and\ \hat{b}$ are unit vectors inclined at an angleθ then prove that

   (i) $\cos\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} + \hat{b}\right|$

   (ii) $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$

   (iii) $\tan\frac{\theta}{2} = \left|\frac{\hat{a} - \hat{b}}{\hat{a} - \hat{b}}\right|$

5. If $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular vectors of equal magnitude. Prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a},\vec{b}$ and $\vec{c}$. Also find angle.

6. For any vector $\vec{a}$ prove that $|\vec{a} \times \hat{\imath}|^2 + |\vec{a} \times \hat{\jmath}|^2 + \left|\vec{a} \times \hat{k}\right|^2 = 2|\vec{a}|^2$

7. Show that $\left(\vec{a} \times \vec{b}\right)^2 = |\vec{a}|^2\left|\vec{b}\right|^2 - \left(\vec{a}.\vec{b}\right)^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

8. If $\vec{a},\vec{b}\ and\ \vec{c}$ are the position vectors of vertices A,B,C of a Δ ABC, show that the area of triangle ABC is $\frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$. Deduce the condition for points $\vec{a},\vec{b}\ and\ \vec{c}$ to be collinear.

9. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c} = 0$ and the angle between b and c is $\pi/6$, prove that $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.

10. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

11. If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{c} = \hat{\jmath} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a}.\vec{b} = 3$.

12. Find the value of $\vec{c}(\vec{a} \times \vec{b}) / |\vec{a} \times \vec{b}|$ if $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$.

13. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$ such that each is perpendicular to sum of the other two, find $|\vec{a} + \vec{b} + \vec{c}|$

14. Decompose the vector $6\hat{\imath} - 3\hat{\jmath} - 6\hat{k}$ in two vectors which are parallel and perpendicular to the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ respectively.

15. If $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, a$\neq 0$, then show that $\vec{b} = \vec{c}$.

16. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$. Prove that $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$

17. Simplify $(\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\}$

18. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find the value of $(\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy$

19. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, find the angle between $\vec{a}$ *and* $\vec{b}$.

20. The magnitude of the vector product of the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ with a unit vector along the sum of the vector $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

21. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, prove that $(\vec{a} - \vec{d})$is parallel to $(\vec{b} - \vec{c})$, where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

22. Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$.

23. Prove that the angle betwen two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

24. If $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

25. Find a unit vector perpendicular to plane ABC, when position vectors of A,B,C are $\widehat{3\imath} - \hat{\jmath} + 2\hat{k}$, $\hat{\imath} - \hat{\jmath} - 3\hat{k}$ and $4\hat{\imath} - \widehat{3\jmath} + \hat{k}$ respectively.

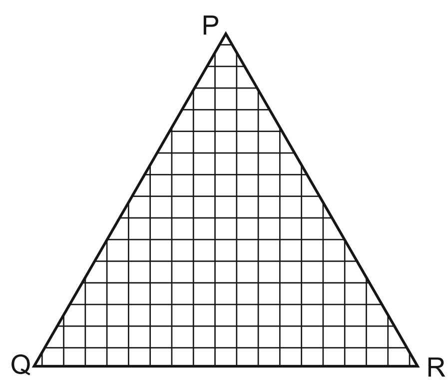
26. Suppose $\vec{a} = \lambda\hat{i} - 7\hat{j} + 3\hat{k}$, $\vec{b} = \lambda\hat{i} + \hat{j} + 2\lambda\hat{k}$. If the angle between $\vec{a}$ and $\vec{b}$ is greater than 90°, then prove that λ satisfies the inequality $-7 < \lambda < 1$.

27. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$ then find the value of $(2\vec{a} - 5\vec{b}).(3\vec{a} + \vec{b})$.

28. Let $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$. Find a vector $\vec{d}$ such that $\vec{a}.\vec{d} = 0$, $\vec{b}.\vec{d} = 2$ and $\vec{c}.\vec{d} = 4$.

Case Study Questions (4 Marks Each)

1. A farmer moves along the boundary of a triangular field PQR. Three vertices of the triangular field are P(2, 1, –2), Q(–1, 2, 1) and R(1, –4, –2) respectively.



On the basis of above information, answer the following questions:

(i) Find the length of PQ.

(ii) Find the $\angle PQR$

(iii) Find the area of the ΔPQR

OR

(iii) Find projection of QP on QR.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT CHOOSE THE CORRECT OPTION.

1. A unit vector perpendicular to both $\hat{i} + \hat{j}$ *and* $\hat{j} + \hat{k}$ is

   (A) $\hat{i} + \hat{j} + \hat{k}$ (B) $\hat{i} - \hat{j} + \hat{k}$

   (C) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$ (D) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

2. If $|\vec{a} \cdot \vec{b}| = 2$, $|\vec{a} \times \vec{b}| = 4$, then the value of $|\vec{a}|^2 |\vec{b}|^2$ is

   (A) 2 (B) 6

   (C) 8 (D) 20

3. The projection of vector $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ on vector $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$ is

   (A) $\frac{9}{19}$ (B) $\frac{9}{\sqrt{19}}$

   (C) $\frac{9}{\sqrt{6}}$ (D) $\frac{19}{9}$

4. If $\vec{a}$ is any vector, then the value of $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$ is

   (A) $|\vec{a}|^2$ (B) $2|\vec{a}|^2$

   (C) $3|\vec{a}|^2$ (D) $4|\vec{a}|^2$

5. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then the angle between $\vec{a}$ and $\vec{b}$ is

   (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

   (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{3}$

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT CHOOSE THE CORRECT OPTION.

1. If $\vec{a}$, $\vec{b}$ and $\vec{a}+\vec{b}$ are unit vectors. Then the value of $|\vec{a}-\vec{b}|$ is

   (A) 0 (B) 1 (C) $\sqrt{2}$ (D) $\sqrt{3}$

2. If $\vec{a}$ and $\vec{b}$ are two vectors *such that* $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$, then the value of $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$ is

   (A) 0 (B) 41 (C) 29 (D) 7

3. If $\vec{c}\cdot(\hat{i}+\hat{j}) = 2, \vec{c}\cdot(\hat{i}-\hat{j}) = 3$ and $\vec{c}\cdot\hat{k} = 0$, then the vector $\vec{c}$ is

   (A) $\frac{1}{2}(5\hat{i}+\hat{j})$ (B) $\frac{1}{2}(5\hat{i}-\hat{j})$

   (C) $\frac{1}{2}(\hat{i}-5\hat{j})$ (D) $\frac{1}{2}(\hat{i}+5\hat{j})$

4. If the projection of $3\hat{i}+\lambda\hat{j}+\hat{k}$ *on* $\hat{i}+\hat{j}$ is $\sqrt{2}$ units, then the value λ is

   (A) 1 (B) –1 (C) 0 (D) 2

5. If $|\vec{a}| = 2, |\vec{b}| = 7$ *and* $\vec{a}\times\vec{b} = 3\hat{i}+2\hat{j}-6\hat{k}$, then the angle between $\vec{a}$ and $\vec{b}$ is

   (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Answers

ONE MARK QUESTIONS

MCQ (1 Mark Each)

1. (c) (7, 3, 0)
2. (b) x = – 1, y = 2
3. (d) $\frac{1}{\sqrt{21}}(4\hat{i}+2\hat{j}-\hat{k})$
4. (a) $\sqrt{\frac{5}{2}}(3\hat{i}+\hat{j})$
5. (a) $\lambda=\frac{16}{5}$
6. (c) 2/3
7. (b) $p=\frac{27}{2}$
8. (c) 0
9. (a) –5
10. (d) $\lambda=\pm2\sqrt{3}$
11. (a) $\frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$
12. (b) 45°
13. (c) $5\sqrt{3}$ sq. units
14. (a) $\lambda = 5$
15. (b) 60°
16. (a) [0, 12]
17. (b) 44
18. (c) $\frac{\pi}{3}$
19. (d) $|\vec{a}|=\sqrt{2}\,|\vec{b}|$
20. (c) $\sqrt{3}$
21. (c)
22. (a)
23. (a)
24. (a)
25. (d)

TWO MARK QUESTIONS

1. $4(\sqrt{2}\hat{i} + \hat{j} + \hat{k})$
2. 22
3. 2
4. $m = 8$
5. 0
6. 3
7. $\lambda = 2$

10. –169

11. $2\sqrt{2}$

12. $\overrightarrow{AC} = 3(\vec{a} - \vec{b})$, $\overrightarrow{BC} = 4(\vec{a} - \vec{b})$

14. $p = -\frac{2}{3}, 2$

15. $\sqrt{33}$

16. $\frac{32}{7}, \frac{32}{49}(2\hat{i} + 6\hat{j} + 3\hat{k})$

17. $x = -\frac{31}{12} \quad y = \frac{41}{12}$

18. $\frac{\pi}{4}$

19. $\frac{\pi}{2}$

THREE MARKS QUESTIONS

1. $x = 3,\ y = 3,\ 1:2$

3. $\vec{d} = 7\hat{\imath} - 7\hat{\jmath} - 7\hat{k}$

5. $\cos^{-1}\frac{1}{\sqrt{3}}$

8. $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

11. $\vec{b} = \frac{5}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$

12. $\frac{4}{\sqrt{38}}$ units

13. $5\sqrt{2}$

14. $\left(-\hat{\imath} - \hat{\jmath} - \hat{k}\right) + \left(7\hat{\imath} - 2\hat{\jmath} - 5\hat{k}\right)$

17. 0

18. 0

19. 60°

20. $\lambda = 1$

22. $\hat{i} - 11\hat{j} - 7\hat{k}$

24. $\vec{\beta} = \left(\frac{3}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}\right) + \left(\frac{1}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} + 3\hat{k}\right)$

25. $\frac{-1}{\sqrt{165}}\left(10\hat{\imath} + 7\hat{\jmath} - 4\hat{k}\right)$

27. $-\frac{11}{2}$

28. $\vec{d} = 2\hat{i} - \hat{j} + \hat{k}$

Case Study Questions

(i) $\sqrt{19}$ units

(ii) $\cos^{-1}\left(\frac{3}{\sqrt{19}}\right)$

(iii) $\frac{7}{2}\sqrt{10}$ square units

OR

(iii) 3 units

SELF ASSESSMENT-1

1. (C) 2. (D)
3. (D) 4. (B)
5. (B)

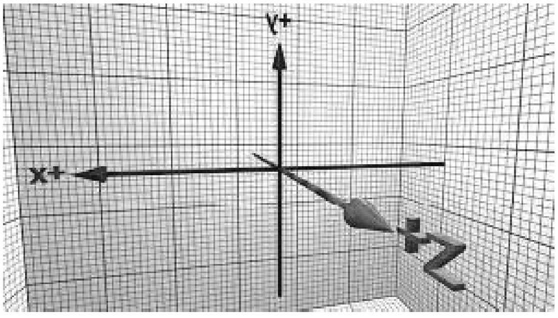
SELF ASSESSMENT-2

1. (D) 2. (A)
3. (B) 4. (B)
5. (A)

CHAPTER 11

THREE-DIMENSIONAL GEOMETRY

In the real world, everything you see is in a three-dimensional shape, it has length, breadth, and height. Just simply look around and observe. Even a thin sheet of paper has some thickness.



Applications of geometry in the real world include the computer-aided design (CAD) for construction blueprints, the design of assembly systems in manufacturing such as automobiles, nanotechnology, computer graphics, visual graphs, video game programming, and virtual reality creation.

The next time you play a mobile game, thank three-dimension geometry for the realistic look to the landscape and the characters that exhibit the game's virtual world.

THREE DIMENSIONAL GEOMETRY

Topics to be covered as per C.B.S.E. revised syllabus (2025-26)

- Direction cosines and direction ratios of a line joining two points.
- Cartesian equation and vector equation of a line.
- Skew lines
- Shortest distance between two lines.
- Angle between two lines.

POINTS TO REMEMBER

- **Distance Formula:** Distance (d) between two points(x_1 , y_1 , z_1)and(x_2, y_2, z_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Section Formula:** line segment AB is divided by P (x, y, z) in ratio m:n

(a) Internally	(b) Externally
$\left(\frac{m\,x_2 + n\,x_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

- **Direction ratio** of a line through $(x_1, y_1, z_1)$and (x_2, y_2, z_2) are$x_2 - x_1, y_2 - y_1, z_2 - z_1$

- **Direction cosines** of a line having direction ratios as a, b, c are:

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Equation of line in space:**

Vector form	Cartesian form
(i) Passing through point $\vec{a}$ and parallel to vector $\vec{b}$; $\vec{r} = \vec{a} + \lambda\vec{b}$	(i) Passing through point (x_1, y_1, z_1) and having direction ratios a, b, c;

	$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$
(ii)Passing through two points $\vec{a}$ and $\vec{b}$; $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$	(ii) Passing through two points $(x_1, y_1 z_1)$ and $(x_2, y_2 z_2)$; $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

- **Angle between two lines:**

Vector form	Cartesian form
(i) For lines $\vec{r}=\overrightarrow{a_1}+\lambda\overrightarrow{b_1}$ and $\vec{r}=\overrightarrow{a_2}+\mu\overrightarrow{b_2}$, $\cos\theta=\frac{\lvert\vec{b}_1\cdot\vec{b}_2\rvert}{\lvert\vec{b}_1\rvert\lvert\vec{b}_2\rvert}$ where 'θ' is the angle between two lines.	(ii) For lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ $\cos\theta=\frac{\lvert a_1a_2+b_1b_2+c_1c_2\rvert}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
(iii) Lines are perpendicular if $\vec{b}_1\cdot\vec{b}_2=0$	(ii) Lines are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$
(iv) Lines are parallel if $\vec{b}_1=k\vec{b}_2$; $k\neq 0$	(i) Lines are parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

- **Shortest distance between two skew lines**

The shortest distance between two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $$d = \left|\frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{\lvert\vec{b}_1 \times \vec{b}_2\rvert}\right|$$ If d = 0, lines are intersecting	The shortest distance between $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}}$$ Where $$D = \{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}$$

- **Shortest distance between two parallel lines**

Let $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ are parallel lines then shortest distance between those lines

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{(\vec{b})}\right| \text{ units}$$

If d = 0, then lines coincident.

Illustration 1:

Are the following lines interesting?

$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$

and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

If yes, find point of intersection.

Solution:

We can write the equations in cartesian form

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2} = l \qquad ...(i)$$

and $$\frac{x-5}{3} = \frac{y+2}{2} = \frac{z}{6} = m \qquad ...(ii)$$

Any point on line (i) P(λ + 3, 2λ +2, 2λ – 4)

Any point on line (ii) Q (3μ + 5, 2μ – 2, 6μ)

Comparing *x*, *y* and *z* coordinate respectively

$\lambda + 3 = 3\mu + 5,\ 2\lambda + 2 = 2\mu - 2,\ 2\lambda - 4 = 6\mu$

or $\lambda - 3\mu = 2,\ 2\lambda - 2\mu = -4,\ 2\lambda - 6\mu = 4$

or $\lambda - 3\mu = 2,\ \lambda - \mu = -2,\ \lambda - 3\mu = 2$

Solving first two, we get $\lambda = -4$, $\mu = -2$

$\because$ $\lambda = -4$, $\mu = -2$, Satisfies $\lambda - 3\mu = 2$

$\therefore$ lines are intersecting

and point of interesting (–1, –6, –12)

Or

Using distance formula

If

th $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 0$

Illustration 2:

Find the foot of perpendicular from the point P(1, 2, –3) to the line $\frac{x+1}{2} = \frac{y-3}{-2} = \frac{z}{-1}$.

Also find the length of the perpendicular and image of P in the given lines.

Solution: We have

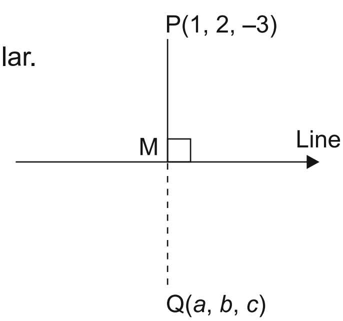
$$\frac{x+1}{2} = \frac{y-3}{-2} = \frac{z}{-1} = \lambda \text{ (say)}$$

$\therefore$ $x = 2\lambda - 1$, $y = -2\lambda + 3$, $z = -\lambda$

Let M($2\lambda - 1, -2\lambda + 3, -\lambda$) be the foot of perpendicular.

DR's of PM are <$2\lambda - 1 - 1, -2\lambda + 3 - 2, -\lambda + 3$>

or < $2\lambda - 2, -2\lambda + 1, -\lambda + 3$>



$\because$ PM is perpendicular to the line

$\therefore$ $2(2\lambda - 2) - 2(-2\lambda + 1) - 1(-\lambda + 3) = 0$

$4\lambda - 4 + 4\lambda - 2 + \lambda - 3 = 0$

$9\lambda - 9 = 0$

$\Rightarrow \lambda = 1$

$\therefore$ Foot of the perpendicular M = (1, 1 – 1)

and PM = $\sqrt{(1-1)^2 + (2-1)^2 + (-3+1)^2} = \sqrt{0+1+4} = \sqrt{5}$

Let Q(a, b, c) be the image of P

As M be the mid point of PQ. (As line is plane mirror)

$\therefore \quad \dfrac{a+1}{2} = 1 \quad \Rightarrow \quad a = 1$

$\dfrac{b+2}{2} = 1 \quad \Rightarrow \quad b = 0$

$\dfrac{c-3}{2} = -1 \quad \Rightarrow \quad c = 1$

$\therefore$ image of P is (1, 0, 1)

ONE MARK QUESTIONS

Multiple Choice Questions (1 Mark Each)

Select the correct option out of the four given options:

1. Distance of the point (a, b, c) from x-axis is

   (a) $\sqrt{b^2 + c^2}$ (b) $\sqrt{c^2 + a^2}$

   (c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 + b^2 + c^2}$

2. Angle between the lines 2x = 3y = – z and 6x = – y = – 4z is

   (a) 45° (b) 60°

   (c) 90° (d) 30°

3. Equation of the line passing through (2, –3, 5) and parallel to

   $\dfrac{x-1}{3} = \dfrac{y-2}{4} = \dfrac{z+1}{-1}$ is

   (a) $\dfrac{x+2}{3} = \dfrac{y-3}{4} = \dfrac{z+5}{-1}$ (b) $\dfrac{x-2}{3} = \dfrac{y-3}{4} = \dfrac{z-5}{1}$

   (c) $\dfrac{x-2}{3} = \dfrac{y+3}{4} = \dfrac{5-z}{1}$ (d) $\dfrac{x-2}{-3} = \dfrac{y+3}{-4} = \dfrac{z-5}{2}$

4. If the lines $\dfrac{x-1}{2} = \dfrac{z-3}{5} = \dfrac{z-1}{\lambda}$ and $\dfrac{z-2}{3} = \dfrac{y+1}{-2} = \dfrac{z}{2}$ are perpendicular, then the value of 'λ' is

   (a) $\lambda = -2$ (b) $\lambda = 2$

   (c) $\lambda = 1$ (d) $\lambda = -1$

5. Cartesian form of line $\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{j} - \hat{k})$ is

(a) $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$ (b) $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z}{-1}$

(c) $\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z}{0}$ (d) $\frac{x+1}{2} = \frac{y+1}{1} = \frac{z}{0}$

6. The coordinates of the foot of the parpendicular drwan from the point (–2, 8 7) on the xz plane is

(a) (0, 8, 0) (b) (–2, 0, 7)

(c) (2, 8, –7) (d) (–2, –8, 7)

7. The length of perpendicular from the point (4, –7, 3) on the y-axis is

(a) 3 units (b) 4 units

(c) 5 units (d) 7 units

8. If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are direction cosimes of a line, then the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is

(a) 1 (b) –1

(c) 2 (d) –2

9. If two lines $x = ay + b$, $z = cy + d$ and $x = a'y + b$, $z = c'y + d$ are perpendicular, then

(a) $aa' + cc' = 1$ (b) $aa' + cc' + 1 = 0$

(c) $\frac{a}{a'} + \frac{c}{c'} = 1$ (d) $\frac{a}{a'} + \frac{c}{c'} + 1 = 0$

10. A point P lines on the line segment joining the points (–1, 3, 2) and (5, 0, 6), if x-coordinate of P is 2, then its z coordinate is

(a) 8 (b) 4

(c) 3 (d) –1

ASSERTION-REASON BASED QUESTIONS

In the following questions a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices:

(a) Both (A) and (R) are true and (R) is the correct explanation of (A)

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A)

(c) (A) is true and (R) is false

(d) (A) is false, but (R) is true

11. Assertion (A) : The vector equation of a line passing through the points (3, 1, 2) and (4, 2, 5) is $\vec{r} = 3\hat{i} + \hat{j} + 2\hat{k} + \lambda(\hat{i} + \hat{j} + 3\hat{k})$

Reason (R) : The vector equation of a line passing through the points with position vector $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

12. Assertion (A) : If a line joining the points (1, 0, 4) and (3, λ, 7) is perpendicular to the line joining the points (1, 2, –1) and (2, 3, 0), then λ = – 5

Reason (R) : Two lines with direction ratios (a_1, b_1, c_1) and $<a_2, b_2, c_2>$ are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

13. Assertion (A) : The coordinates of the point where the line

$\vec{r} = (3\hat{i} + \hat{j} - \hat{k}) + \lambda(-\hat{i} + 2\hat{j} + 3\hat{k})$ cuts xy-plane and $\left(\frac{8}{3}, \frac{-5}{3}, 0\right)$

Reason (R) : The z-coordinate of any point on xy-plane is 0.

14. Assertion (A) : Lines $\frac{x+1}{-1} = \frac{2-y}{-2} = \frac{z-3}{3}$ and $\frac{2-x}{-3} = \frac{y-1}{4} = \frac{z+2}{-1}$ intersect at a point.

Reason (R) : Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are intersecting if $(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) \neq 0$.

TWO MARKS QUESTIONS

1. Find the equation of a line passing though (2, 0, 5) and which is parallel to line $6x - 2 = 3y + 1 = 2z - 2$

2. The equation of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line

3. If a line makes angle α, β, γ with Co-ordinate axis then what is the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

4. Find the equation of a line passing through the point (2, 0, 1) and parallel to the line whose equation is $\vec{r} = (2\lambda + 3)\hat{i} + (7\lambda - 1)\hat{j} + (-3\lambda + 2)\hat{k}$

5. Find the condition that the lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ may be perpendicular to each other.

6. Show that the lines $x = -y = 2z$ and $x + 2 = 2y - 1 = -z + 1$ are perpendicular to each other.

7. Find the equation of the line through (2, 1, 3) and parallel to the line $\frac{2x-1}{2}=\frac{4-y}{7}=\frac{z+1}{2}$ in cartesian and vector form.

8. Find the cartesian and vector equation of the line through the points (2, –3, 1) and (3, –4, –5)

9. For what value of λ and μ the line joining the points (7, λ, 2), (μ, –2, 5) is parallel to the line joining the points (2, –3, 5), (–6, –15, 11)?

10. If the points (–1, 3, 2), (–4, 2, –2) and (5, 5, λ) are Collinear, find the value of λ.

THREE/FIVE MARKS QUESTIONS

1. Find vector and Cartesian equation of a line passing through a point with position vector $2\hat{\imath}-\hat{\jmath}+\hat{k}$ and which is parallel to the line joining the points with position vectors $-\hat{\imath}+4\hat{\jmath}+\hat{k}$ and $\hat{\imath}+2\hat{\jmath}+2\hat{k}$.

2. Find image (reflection) of the point (7, 4, –3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

3. Show that the lines line $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and line $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect each other. Find the point of intersection.

4. Find the shortest distance between the lines:

$$\overline{r} = \hat{\imath}+2\hat{\jmath}+3\hat{k}+\mu(2\hat{\imath}+3\hat{\jmath}+4\hat{k}) \text{ and}$$

$$\overline{r} = (2\hat{\imath}+4\hat{\jmath}+5\hat{k})+\lambda(3\hat{\imath}+4\hat{\jmath}+5\hat{k}).$$

5. Find shortest distance between the lines:

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{5-y}{2} = \frac{z-7}{1}$

6. Find the shortest distance between the lines:

$\overline{r} = (1-\lambda)\hat{\imath} + (\lambda - 2)\hat{\jmath} + (3 - 2\lambda)\hat{k}$

$\overline{r} = (\mu + 1)\hat{\imath} + (2\mu - 1)\hat{\jmath} - (2\mu + 1)\hat{k}$

7. Find the foot of perpendicular from the point $2\hat{\imath} - \hat{\jmath} + 5\hat{k}$ on the line $\overline{r} = \left(11\hat{\imath} - 2\hat{\jmath} - 8\hat{k}\right) + \lambda\left(10\hat{\imath} - 4\hat{\jmath} - 11\hat{k}\right)$. Also find the length of the perpendicular.

8. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonal of a cube. Prove that $cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \frac{4}{3}$

9. Find the length and the equations of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

10. Show that $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{2}, z = 2.$ do not intersect each other.

11. If the line $\frac{x+2}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of *k*.

12. Find the equation of the line which intersects the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4}$ and passes through the point (1, 1, 1).

13. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angle of $\pi/3$.

14. Find the foot of perpendicular drawn from the point (2, –1, 5) to the line

$\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$

Also find the length of the perpendicular. Hence find the image of the point (2, –1, 5) in the given line.

15. Find the image of the point P(2, –1, 11) in the line

$\vec{r} = (2\hat{i} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$

16. Find the point(s) on the line through the point P(3, 5, 9) and Q(1, 2, 3) at a distance 14 units from the mid-point of segment PQ.

17. Find the shortest distance between the following pair of lines

$\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{1}; z = 2$

Hence write whether the lines are intersecting or not.

18. Find the foot of perpendicular from the point (1, 2, 3) to the line

$\vec{r} = (6\hat{i} + 7\hat{j} + 7\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$

Also find the equation of the perpendicular and length of perpendicular.

19. Find the equation of the line passing through (–1, 3, –2) and perpendicular to the lines $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+5}{3}$ and $\frac{x-2}{-3} = \frac{y}{2} = \frac{z+1}{5}$

20. Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{3-z}{-2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3)

21. The points P(4, 5, 10), Q(2, 3, 4) and R(1, 2, –1) are three vertices of a parallelogram PQRS. Find the vector equations of the sides PQ and QR and also find the coordinates of point S.

22. Find the equation of perpendicular from the point (3, –1, 11) to the line

$\frac{x}{2} = \frac{2y-4}{6} = \frac{3-z}{-4}$.

Also find the foot of the perpendicular and the length of the perpendicular.

23. Show that the lines $\frac{1-x}{-2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{-1+y}{2} = z$ are intersecting. Also find the point of intersection.

24. For what value of 'λ', the following are Skew lines?

$$\frac{x-4}{5}=\frac{1+y}{2}=z, \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-\lambda}{4}$$

25. Find the vector equation of the line passing through (2, 1, –1) and parallel to the line $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ Also find the distance between these two lines.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. The foot of perpendicular drawn from the point (2, –1, 5) to the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$ is

   (a) (2, 1, 3) (b) (3, 1, 2)

   (c) (1, 2, 3) (d) (3, 2, 1)

2. The shortest distance between the lines $\vec{r}=(6\hat{i}+2\hat{j}+2\hat{k})+\lambda(\hat{i}-2\hat{j}+2\hat{k})$ and $\vec{r}=(-4\hat{i}-4\hat{k})+\mu(3\hat{i}-2\hat{j}-2\hat{k})$ is

   (a) 10 units (b) 9 units

   (c) 12 units (d) 9/2 units

3. If the x-coordinate of a point A on the join of B(2, 2, 1) and C(5, 1, –2) is 4 then its z-coordinate is

   (a) –2 (b) –1

   (c) 1 (d) 2

4. The distance of the point M(a, b, c) from the x-axis is

   (a) $\sqrt{b^2+c^2}$ (b) $\sqrt{c^2+a^2}$

   © $\sqrt{a^2+b^2}$ (d) $\sqrt{a^2-b^2+c^2}$

5. The straight line $\frac{x-3}{3}=\frac{y-2}{1}=\frac{z-1}{0}$ is

   (a) parallel to x-axis (b) parallel to y-axis

   (c) parallel to z-axis (d) perpendicular to z-axis

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. The shortest distance between the line $\frac{x-3}{3}=\frac{y}{0}=\frac{z}{-4}$ and y-axis is

(a) $\frac{12}{5}$ units (b) $\frac{1}{5}$ units

(c) 0 units (d) 3 units

2. The point of intersection of the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ is

(a) $\left(\frac{1}{3},\frac{-1}{3},-\frac{2}{3}\right)$ (b) $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

(c) $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},\frac{-1}{2},\frac{-3}{2}\right)$

3. If a line makes the same angle α, with each of the x and z axes and the angle β with y-axis such that $3\sin^2\alpha = \sin^2\beta$, then the value of $\cos^2\alpha$ is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{2}{3}$

4. If the lines $\frac{x+3}{k-5}=\frac{y-1}{1}=\frac{5-z}{-2k-1}$ and $\frac{x+2}{-1}=\frac{2-y}{-k}=\frac{z}{5}$ are perpendicular, then the value of k is

(a) 1 (b) –1

(c) 2 (d) –2

5. The image of the point P(-1, 8, 4) to the line $\frac{x}{2}=\frac{y+1}{-2}=\frac{z-3}{-4}$ is

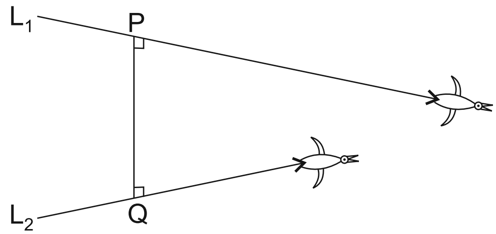
(a) (5, 4, 4) (b) (5, 0, 4)

(c) (-3, -6, 10) (d) (1, 8, 4)

Case Study Based Questions

1. Two birds are flying in the space along straight path L_1 and L_2

   (Neither parallel nor intersecting) where,



$$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

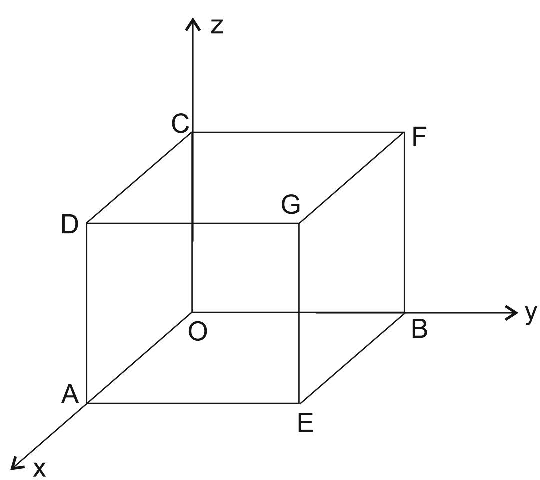
$$L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

P and Q are the points on the path L1 and L2 respectively such that PQ is perpendicular on both paths L1 and L2. On the basis of above information, answer the following questions

(i) Find the length PQ

(ii) Find the equation of PQ

2. A carpenter designed a Cuba of side a units and put it in 3 dimensional system such that one vertex at origin and adjacent sides on three coordinate axes as shown in figure



Based on the above information, answer the following questions:

(i) Write the coordinates of the vertices D, E, F and G.

(ii) Find the direction ratios of the diagonal OG.

(iii) Find the direction cosines of the diagonals CE and DB

OR

(iii) Find the angle between CE and DB.

ANSWERS

ONE MARK QUESTIONS

1. (a) $\sqrt{b^2 + c^2}$
2. (c) 90°
3. (c) $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-5}{-1}$
4. (b) $\lambda = 2$
5. (a) $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$
6. (b) (–2, 0, 7)
7. (c) 5 units
8. (b) –1
9. (b) aa′ + cc′ + 1 = 0
10. (b) 4
11. (a)
12. (b)
13. (d)
14. (c)

TWO MARK QUESTIONS

1. $\frac{x-2}{1} = \frac{y}{2} = \frac{z-5}{3}$
2. $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$
3. 2
4. $\vec{r} = (2\hat{i} + \hat{k}) + \lambda(2\hat{i} + 7\hat{j} - 3\hat{k})$
5. aa′ + cc′ + 1 = 0
7. $\frac{x-2}{1} = \frac{y-1}{-7} = \frac{z-3}{2}$,
   $\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(\hat{i} - 7\hat{j} + 2\hat{k})$
8. $\frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6}$,
   $\vec{r} = (2\hat{i} - 3\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} - 6\hat{k})$
9. $\lambda = 4$
   $\mu = 3$
10. $\lambda = 10$

THREE/FIVE MARK QUESTIONS

1. $\bar{r} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) + \lambda(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$

2. $\left(-\frac{51}{7}, -\frac{18}{7}, \frac{43}{7}\right)$

3. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

4. $\frac{1}{\sqrt{6}}$

5. $2\sqrt{29}$ units

6. $\frac{8}{\sqrt{29}}$

7. $(1, 2, 3), \sqrt{14}$

9. $SD = 14\, units,\ \frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$

11. K = 12

12. $\frac{x-1}{3} = \frac{y-1}{10} = \frac{z-1}{17}$

13. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

14. (1, 2, 3), $\sqrt{14}$, (0, 5, 1)

15. (6, 7, 3)

16. $\left(6, \frac{19}{2}, 18\right)$, $\left(-2, \frac{-5}{2}, -6\right)$

17. $\frac{9}{\sqrt{195}}$, Not intersecting

18. (3, 5, 9), $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$, 7 units

19. $\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$

20. $(-2, -1, 3), \left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right)$

21. $PQ: \vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + 6\hat{k})$
$QR: \vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \mu(\hat{i} + \hat{j} + 5\hat{k})$, Point S(3,4,5)

22. $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$, (2,5,7), $\sqrt{53}$ units

23. (–1, –1, –1)

24. $\lambda \neq 69/11$

25. $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(2\hat{i} - \hat{j} + \hat{k})$, $\sqrt{\frac{11}{6}}$ units

SELF ASSESSMENT TEST-1

1. (C) 2. (B) 3. (B) 4. (A) 5. (D)

SELF ASSESSMENT TEST-2

1. (A) 2. (D) 3. (C) 4. (B) 5. (C)

CASE STUDY BASED QUESTIONS

1. (i) $3\sqrt{30}$ units (ii) $\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 5\hat{j} - \hat{k})$

2. (i) D(a, 0, a), E(a, a, 0), F(0, a, a) & G(a, a, a)

(ii) <1, 1, 1>

(iii) Direction cosines of CE are $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

DB are $\left\langle \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$

OR

Angle between CE and DB = $\cos^{-1}\left(\frac{1}{3}\right)$

CHAPTER-12

LINEAR PROGRAMMING

Linear programming is used to obtain optimal solutions for operations research. Using LPP, researchers find the **best**, most economical **solution** to a problem within all of its **limitations**, or constraints.

Few examples of applications of LPP

(i) **Food and Agriculture:** In nutrition, Linear programming provides a powerful tool to aid in planning for dietary needs. Here, we determine the different kinds of foods which should be included in a diet so as to **minimize** the cost of the desired diet such that it contains the minimum amount of each nutrient.

(ii) **Transportation:** Systems rely upon linear programming for cost and time efficiency.

Airlines use linear programming to optimize their profits according to different seat prices and customer demand. Because of this only, efficiency of airlines increases and expenses are decreased.

TOPICS TO BE COVERED AS PER CBSE LATEST CURRICULUM 2025-26

- Introduction, constraints, objective function, optimization.
- Graphical method of solution for problems in two variables.
- Feasible and infeasioble region (bounded or unbounded)
- Feasible and infeasible solutions.
- Optimal feasible solutions (upto three non-trival constraints)

KEY POINTS :

- **OPTIMISATION PROBLEM :** It is a problem which seeks to maximize or minimize a function. An optimisation problem may involve maximization of profit, minimization of transportation cost etc, from available resources.
- **A LINEAR PROGRAMMING PROBLEM (LPP) :** LPP deals with the optimisation (maximisation/minimisation) of a linear function of two variables (say x and y) known as objective function subject to the conditions that the variables are non negative and satisfy a set of linear inequalities (called linear constraints). A LPP is a special type of optimisation problem.
- **OBJECTIVE FUNCTION :** Linear function $z = ax + by$ where a and b are constants which has to be maximised or minimised is called a linear objective function.
- **DECISION VARIABLES :** In the objective function $z = ax + by$, x and y are called decision variables.
- **CONSTRAINTS :** The linear inequalities or restrictions on the variables of an LPP are called constraints.

  The conditions $x \geq 0, y \geq 0$ are called non-negative constraints.
- **FEASIBLE REGION :** The common region determined by all the constraints including non-negative constraints $x \geq 0, y \geq 0$ of a LPP is called the feasible region for the problem.
- **FEASIBLE SOLUTION :** Points within and on the boundary of the feasible region for a LPP represent feasible solutions.
- **INFEASIBLE SOLUTIONS :** Any point outside the feasible region is called an infeasible solution.
- **OPTIMAL (FEASIBLE) SOLUTION :** Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.
- **THEOREM 1 :** Let R be the feasible region (convex polygon) for a LPP and let $z = ax + by$ be the objective function. When z has an optimal value (maximum or minimum), where x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.
- **THEOREM 2 :** Let R be the feasible region for a LPP. & let $z = ax + by$ be the objective function. If R is bounded, then the objective function z has both a maximum and a minimum value on R and each of these occur at a corner point of R.

  If the feasible region R is unbounded, then a maximum or minimum value of the objective function may or not exist. However, if it exists it must occur at a corner point of R.

- **MULTIPLE OPTIMAL POINTS :** If two corner points of the feasible region are optimal solutions of the same type i.e both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

Illustration:

A company produces two types of belts A and B. Profits on these belts are Rs. 2 and Rs. 1.50 per belt respectively. A belt of type A requires twice as much time as belt of type B. The company can produce atmost 1000 belts of type B per day. Material for 800 belts per day is available. Atmost 400 buckles for belts of type A and 700 for type B are available per day. How much belts of each type should the company produce so as to maximize the profit?

Solution: Let the company produces x no. of belts of type A and y no. of belts of type B to maximize the profit.

$\therefore$ **Objective function** Max $z = 2x + 1.5y$

As, maximum 1000 belts of type B : 1 day

$\therefore$ 1 belt of type B : $\left(\frac{1}{1000}\right)^{th}$ of a day

ATQ, 1 belt of type A : $\left(\frac{2}{1000}\right)^{th}$ of a day

$\therefore \quad \frac{2x}{1000} + \frac{y}{1000} \le 1$

$\Rightarrow \quad 2x + y \le 1000$

L.P.P becomes

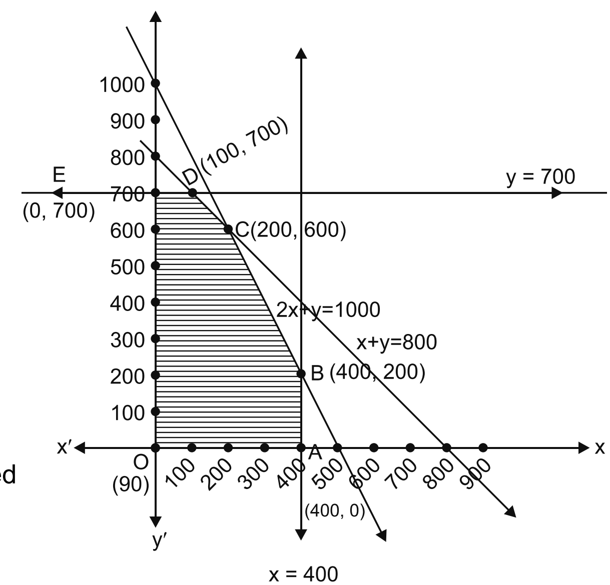
Max $z = 2x + 1.5y$

s.t. $2x + y \le 1000$

$x + y \le 800$

$x \le 400$, $y \le 700$, $x \ge 0$, ≥ 0

Here, the feasible negion is bounded given be region OABCDE.



Using Corner point method.

Corner Points	Obj. fn. $z = 2x + 1.5y$
O (0, 0)	0
A (400, 0)	800
B (400, 200)	1100
C (200, 600)	1300 — max z.
D (100, 700)	1250
E (0, 700)	1050

$\therefore$ Optimal solution is given by C(200, 600)

i.e. company should produce 200 belts of type A and 600 belts of type B so as to maximize the profit of Rs. 1300.

ONE MARK QUESTIONS

1. The solution set of the inequation $3x + 4y < 7$ is:

   (a) Whole xy plane except the points lying on the line $3x + 5y = 7$

   (b) Whole xy plane alogn with the points lying on the line $3x + 5y = 7$

   (c) Open half plane containging the origin except the point of line $3x + 5y = 7$

   (d) Open half plane not containging the origin except the point of line $3x + 5y = 7$

2. Which of the following points solisfies both the inequations $2x + y \leq 10$ and $x + 2y \geq 8$?

   (a) (–2, 4) (b) (3, 2)

   (c) (–5, 6) (d) (4, 2)

3. The objective function $Z = ax + by$ of LPP has maximum value 42 at (4, 6) and minimum value 19 at (3, 2). Which of the following is true?

   (a) $a = 9, b = 1$ (b) $a = 5, b = 2$

   (c) $a = 3, b = 5$ (d) $a = 5, b = 3$

4. The corner points of the feasible region of a LPP are (0, 4), (7, 0) and $\left(\frac{20}{3}, \frac{4}{3}\right)$. If $z = 30x + 24y$ is the objective functions, then (maximum value of z-minimum value of z) is equal to

   (a) 40 (b) 96

   (c) 120 (d) 136

5. The minimum value of $z = 3x + 8y$ subject to the constraints $x \leq 20$, $y \geq 10$ and $x \geq 0$, $y \geq 0$ is

   (a) 80 (b) 140

   (c) 0 (d) 60

6. The number of corner points of the feasible region determined by the constraints

   $x - y \geq 0$, $2y \leq x + 2$, $x \geq 0$, $y \geq 0$ is

   (a) 2 (b) 3

   (c) 4 (d) 5

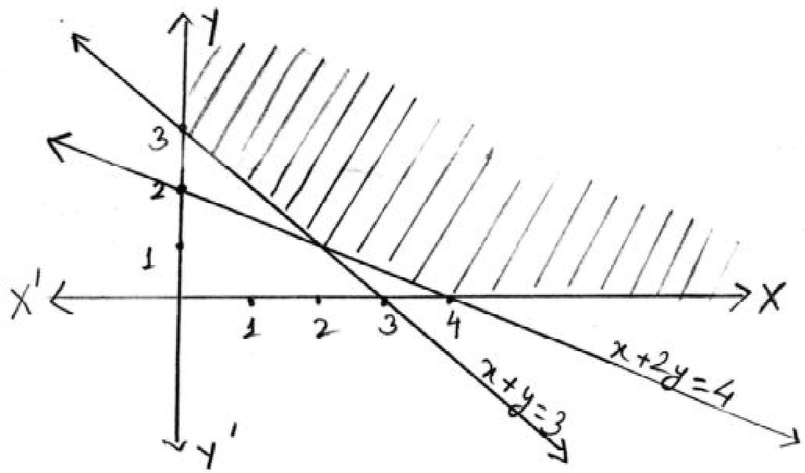
7. The no. of feasible solutions of the L.P.P. given as maximise $z = 15x + 30y$ subject the constraints:

   $3x + y \leq 12$, $x + 2y \leq 10$, $x \geq 0$, $y \geq 0$ is

   (a) 1 (b) 2

   (c) 3 (d) infinite

8. The feasible region of a linear programming problem is shown in the figure below:



Which of the following are the possible constraints?

(a) $x + 2y \geq 4, x + y \leq 3, x \geq 0, y \geq 0$

(b) $x + 2y \leq 4, x + y \leq 3, x \geq 0, y \geq 0$

(c) $x + 2y \geq 4, x + y \geq 3, x \geq 0, y \geq 0$

(d) $x + 2y \geq 4, x + y \leq 3, x \leq 0, y \leq 0$

9. L.P.P. is a process of finding

(a) Maximum value of the objective function

(b) Minimum value of the objective function

(c) Optimum value of the objective function

(d) None of these

10. Which of the following statements is correct?

(a) Every L.P.P. admits an optimal solution

(b) A L.P.P. admits a unique optimal solution

(c) If a L.P.P. admits two optimal solutions, it has an infinite number of optimal solutions

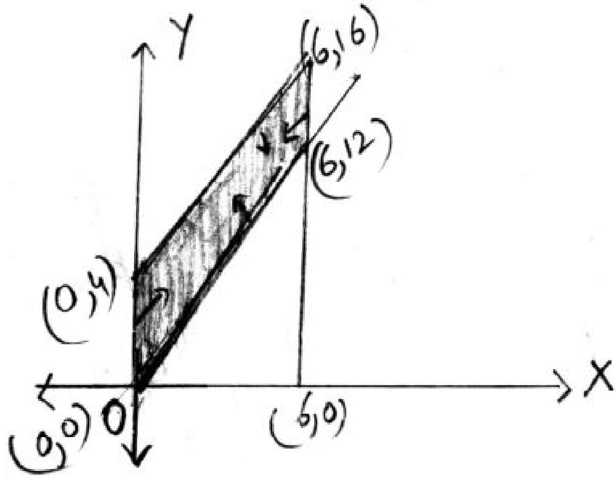
(d) The set of all feasible solution of a L.P.P. is not a convex set

11. Region represented by $x \geq 0$, $y \geq 0$ is

(a) First quadrant (b) Second quadrant

(c) Third quadrant (d) Fourth quadrant

12. The feasible region for L.P.P. is shown shaded in the figure. Let $f = 3x - 4y$ be the objective function, then maximum value of f is



(a) 12 (b) 8

(c) 0 (d) –18

13. The area of the feasible region fo the following constraints

$3y + x \geq 3$, $x \geq 0$, $y \geq 0$ will be

(a) Bounded (b) Unbounded

(c) Convex (d) Concave

14. The line $5x + 4y \geq 20, x \leq 6, y \leq 4$ form,

(a) A square (b) A rhombus

(c) A triangle (d) A qudrilateral

15. The graph of inequations $x \leq y$ and $y \leq x + 3$ is located in

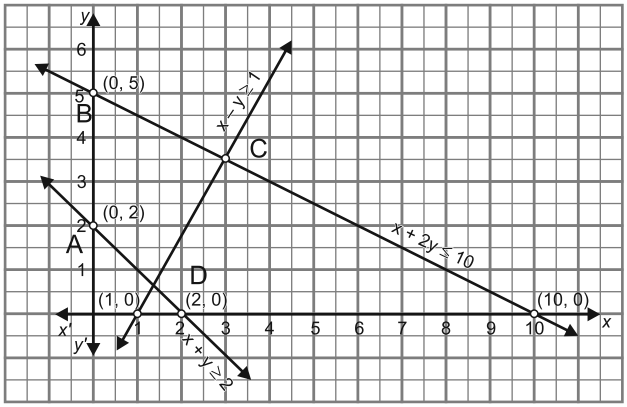
(a) II quadrant (b) I, II quadrant

(c) I, II and III quadrant (d) II, III, IV quadrant

16. The corner points of the bounded feasible region determined by a system of linear constraints are (0, 3). Let Z = px + qy where p, q > 0. Then condition on p and q so that the minimum of Z occurs at (1, 1) and (3, 0) is :

(a) P = 2q (b) p = q/2 (c) p = 3q (d) p = q

17. The feasible region ABCD corresponding to the linear constraints of a linear programming problem is given below :

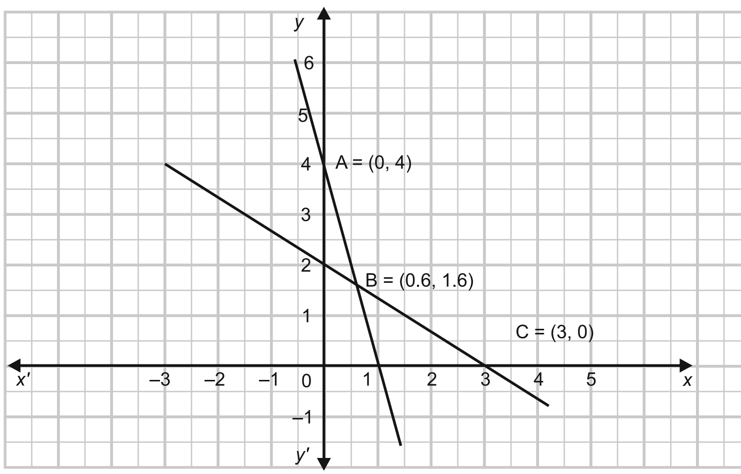


Which of the following is not a constraint to the given linear programming problem?

(a) $x + y \geq 2$ (b) $x + 2y \leq 10$ (c) $x - y \geq 1$ (d) $x - y \leq 1$

18. The solution set of the inequality $3x + 5y < 4$ is
(a) An open half – plane not containing the origin.
(b) An open half – plane containing the origin.
(c) An whole XY – plane not containing the line $3x + 5y = 4$
(d) A close half – plane containing the origin.

19. The corner points of the shaded unbounded feasible region of an LPP are (0, 4), (0, 6, 1) and (3, 0) as shown in the figure. The minimum value of the objective function $Z = 4x + 6y$ occurs at



(a) (0.6, 1.6) (b) (3, 0) only (c) (0.6, 1.6) and (3, 0) only
(d) At every point of the line segment joining the points (0.6, 1.6) and (3, 0)

20. The corner points of the bounded feasible region determined by the system of linear constraints are (0, 0), (4, 0), (2, 4) and (0, 5). If the maximum value of $z = ax + by$
where $a, b > 0$ occurs at both (2, 4) and (4, 0) then
(a) $a = 2b$ (b) $2a = b$ (c) $a = b$ (d) $3a = b$

ASSERTION-REASON TYPE QUESTIONS

Directions: Each of these questions contains two statements, Assertion (A) and Reason (R). Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A)

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A)

(c) (A) is true and (R) is false

(d) (A) is false but (R) is ture

21. Assertion (A) : If a L.P.P. admits two optional solution then it has infinitely many optimal solution.

    Reason (R) : If the value of the objective function of a L.P.P. is same at two corners then it is same at every point on the line segment joining the two corner pionts.

22. Assertion (A): The solution region satisfied by the inequalities $x + y \leq 5$, $x \leq 4$, $y \leq 4$, $x \geq 0$, $y \geq 0$ is bounded.

    Reason (R) : A region in x-y plane is said to be bounded if it can be enclosed within a circle.

23. Assertion (A) : Minimize $z = x^2 + 2xy + y^2$ can be considered as the objective function for the L.P.P.

    Reason (R) : Objective function of the L.P.P. is of this type $z = ax + by$; a and b are real numbers i.e. z is linear function of x and y.

24. Assertion (A) : The region represented by the inequalities $x \geq 6$, $y \geq 2$, $2x + y \geq 10$, $x \geq 0$, $y \geq 0$ is empty.

    Reason (R) : There is no (x, y) that satisfies all the constraints.

25. Assertion (A) : Corner points of the feasible region for an L.P.P. are (0, 2), (3, 0), (6, 0), (6, 8) and (0, 5). Let $F = 4x + 6y$ be the objective function. The minimum value of F occurs at (0, 2) only.

    Reason (R) : Minimum value of F occurs at all the infinite no. points that lie on the line segment joining (0, 2) and (3, 0).

THREE MARKS QUESTIONS

1. Solve the following linear programming problem graphically:

   Maximise $z = -3x - 5y$

   subject to the constraints

   $-2x + y \leq 4$

   $x + y \geq 3$

   $x - 2y \leq 2$

   $x \geq 0, y \geq 0$

2. Solve the following LPP graphically:

Maximise $z = 5x + 3y$

s.t. the constraints

$3x + 5y \leq 15$

$5x + 2y \leq 10$

$x, y \geq 0$

3. Solve the following LPP graphically

Maximise $z = x + 2y$

s.t. $x + 2y \geq 100$

$2x - y \leq 0$

$2x + y \leq 200$

$x \geq 0, y \geq 0$

4. The objective function $z = 4x + 3y$ of a LPP under some constraints is to be maximized and minimized. The corner points of the feasible region are A(0, 700), B(100, 700), C(200, 600) and D(400, 200). Find the point at which z is maximum and the point at which z is minimum. Also find the corresponding maximum and minimum values of z.

5. Solve graphically

Minimise : $z = -3x + 4y$

s.t. $3x + 2y \leq 12$

$x, y \geq 0$

6. Solve the following LPP graphically

Minimise: $Z = 60x + 80y$

s.t. $3x + 4y \geq 8$

$5x + 2y \geq 11$

$x, y \geq 0$

7. Solve graphically

Maximise : $z = 600x + 400y$

s.t. $x + 2y \leq 12$

$2x + y \leq 12$

$x + 1.25y \geq 5$

$x, y \geq 0$

8. Solve graphically

Maximise : $P = 100x + 5y$

s.t. $x + y \leq 300$

$3x + y \leq 600$

$y \leq x + 200$

9. Solve the LPP graphically

Minimize $z = 5x + 10y$

s.t. $x + 2y \leq 120, x + y \geq 60, x - 2y \geq 0$

$x \geq 0, y \geq 0$

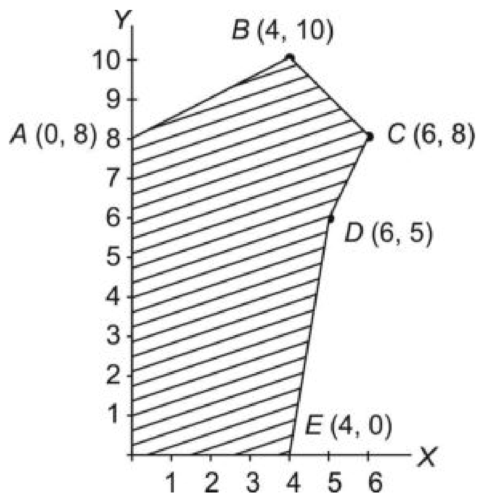
10. Determine graphically the minimum value of the following objective function:
$z = 500x + 400y$
s.t. $x + y \leq 200$
$x \geq 20$
$y \geq 4x$
$y \geq 0$

11. Find graphically, the maximum value of $Z = 2x + 5y$, subject to the constraints given below:
$x + 2y < 4$, $3x + y \leq 6$, $x + y \leq 4$, $x \geq 0$, $y \geq 0$

FIVE MARKS QUESTIONS

Q. 1 Solve the following LPP graphically.
Maximize $z = 3x + y$ subject to the constraints
$x + 2y \geq 100$
$2x - y \leq 0$
$2x + y \leq 200$
$x, y \geq 0$

Q.2 The corner points of the feasible region determined by the system of linear constraints are as shown below.



Answer each of the following :

(i) Let $z = 3x - 4y$ be the objective function. Find the maximum and minimum value of z and also the corresponding points at which the maximum and minimum value occurs.

(ii) Let $z = px + qy$ where $p, q > 0$ be the objective function. FInd the condition on p and q so that the maximum value of z occurs at B (4, 10) and C (5, 8). Also mention the number of optimal solutions in this case.

CASE STUDY QUESTIONS

Q. 1 A man rides his motorcycle at the speed of 50 km/hr. He has to spend Rs 2/km on petrol. But if he rides it at a faster speed of 80 km/hr, the petrol cost increases to Rs 3/km. He has atmost Rs 120 to spend on petrol and one hr's time. he wishes to find the maximum distance that he can travel.

Based on the above information answer the following questions.

(1) If he travels x km with the speed of 50 km/hr and y km with the speed of 80 km/hr, then write the objective function

(2) Find the Maximum distance man can travel?

Q.2 Two tailors A and B earn Rs 150 and Rs 200 per day respectively. A can sticth 6 shirts and 4 pants per day, while B can stitch 10 shirts and 4 pants per day. it is desired to produce atleast 60 shirts and 32 pants at a minimum labour cost.

Tailor A **Tailor B**

Based on the above information answer the following.

(1) If x and y are the number of days A and B work respectively then find the objective function for this LPP

(2) Find the optimal solution for this LPP and the minimum labour cost?

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Objective functin of a L.P.P. is
   - (a) A constraint
   - (b) A function of be optimised
   - (c) A relation between the variables
   - (d) None of these

2. The solution set of the inequation $2x + y > 5$ is
   - (a) Open half plans that contains the origion
   - (b) Open half plane not containing the origin
   - (c) Whole xy-plane except the points lying on the line $2x + y = 5$
   - (d) None of these

3. Which of the following statements is correct?
   - (a) Every L.P.P admits an optimal solution
   - (b) A L.P.P. admits unique optimal solution
   - (c) If a LPP admits two optimal solutions, it has an infinite number of optimal solutions
   - (d) None of these

4. Solution set of inequation $x \geq 0$ is
   - (a) Half plane on the left of y-axis
   - (b) Half plane on the right of y-axis excluding the points on y-axis
   - (c) Half plane on the right of y-axis including the points on y-axis
   - (d) None of these

5. In a LPP, the constraints on the decision variables x and y are

   $x - 3y \geq 0$, $y \geq 0$, $0 \leq x \leq 3$.

   The feasible region
   - (a) is not in the first quadrant
   - (b) is bounded in the first quadrant
   - (c) is unbounded in the first quadrant
   - (d) doesn't exist

SELF ASSESSMENT-2

EAH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Solution set of the inequation $y \leq 0$ is
   (a) Half plane below the x-axis excluding the points on x-axis
   (b) Half plane below the x-axis including the points on x-axis
   (c) Half plane above the x-axis
   (d) None of these
2. Regions represented by inequations $x \geq 0$, $y \geq 0$ is
   (a) first quadrant
   (b) second quadrant
   (c) third quadrant
   (d) fourth quadrant
3. The feasible region for an LPP is always
   (a) concavo convex polygen
   (b) concave poloygon
   (c) convex polygon
   (d) None of these
4. If the constraints in a linear programming problem are changed then
   (a) the problem is to be reevaluated
   (b) solution not defined
   (c) the objective function has to be modified
   (d) the change in constraints is ignored
5. L.P.P. is as follows:

   Minimize $Z = 30x + 50y$

   Subject to the constraints,

   $3x + 5y \geq 15$

   $2x + 3y \leq 18$

   $x \geq 0, y \geq 0$

   In the feasible region, the minimum value of Z occurs at
   (a) a unique point
   (b) no point
   (c) infinitely many points
   (d) two points only

ANSWER

One Marks Questions

1. (c)
2. (d) (4, 2)
3. (c) $a = 3$, $b = 5$
4. (d) 136
5. (a) 80
6. (a) 2
7. (d) infinite
8. (c)
9. (c)
10. (c)
11. (a)
12. (c) 0
13. (b) unbounded
14. (c) A triangle
15. (c)
16. (b)
17. (c)
18. (b)
19. (d)
20. (b)
21. (a)
22. (a)
23. (d)
24. (a)
25. (d)

Three Marks Questions

1. Optimal solution $\left(\frac{8}{3}, \frac{1}{3}\right)$, maximize = $\frac{-29}{3}$ feasible region unbounded.
2. Optimal solution $\left(\frac{20}{19}, \frac{45}{19}\right)$, maximize = $\frac{235}{19} = 12.3$
3. Optimal solution (0, 200), maximize = 400
4. Maximize z = 2600 at C(200, 600) and minimize z is 2100 at A(0, 700)
5. Minimize z = – 12 at (4, 0)
6. Unbounded, minimize z = 160. It occurs at all the points onthe line segment joining $\left(2, \frac{1}{2}\right)$ and $\left(\frac{8}{3}, 0\right)$. So, infinite optimal solutions.
7. Maximize z = 4000 at (4, 4)
8. Maximize z = 20,000 at (200,0)
9. Miximize z = 300 at (60, 0)
10. Miximize z =42000 at (20, 80)
11. Max Z = 4 at (2,0)

Five Marks Questions

1. Max z = 250 at x = 50, y = 100
2. (i) Max z = 12 at (4,0) and min z = –32 at (0,8)

   (ii) P = 2q, infinite solutions lying on the line segment joining the points B and C

CASE STUDIES QUESTIONS

1. (i) maximize z = x + y (ii) $54\frac{2}{7}$ km

2. (i) minimize z = 150x + 200y (ii) (5, 3) and Rs. 1350

SELF ASSESSMENT-1

1. (b)	2. (b)	3. (c)	4. (c)	5. (b)

SELF ASSESSMENT-2

1. (b)	2. (a)	3. (c)	4. (a)	5. (c)

CHAPTER-13

PROBABILITY

Probability is the branch of mathematics that deals with assigning a numerical quantity ($0 \leq p \leq 1$) to the happening/non happening of any event.

Real Life Applications
- Weather Forecasting
- Sports betting
- Investing in stock market
- Politics
- How likely a natural disaster like earthquake, hurricane etc. will strike the country in a given year.

A sports betting company may look at the current record of two teams A and B and determine which team has higher probability of winning and do the sports betting accordingly.

TOPICS TO BE COVERED AS PER CBSE LATEST CURRICULUM (2025-26)

- Conditional probability
- Multiplication theorem on probability
- Independent events
- Total probability and Baye's theorem

KEY POINTS

Conditional Probability : If A and B are two events associated with the same sample space of a random experiment, then the conditional probability of the event A under the condition that the event B has already occurred, written as $P(A|B)$, is given by

$$P(A|\text{B}) = \frac{(P \cap B)}{P(B)}, \quad P(B) \neq 0.$$

Properties :

(1) $P(S|F) = P(F|F) = 1$ where S denotes sample space

(2) $P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$

(3) $P(E'|F) = 1 - P(E|F)$

Multiplication Rule : Let E and F be two events associated with a sample place of an experiment. Then

$$P(E \cap F) = P(E)\,P(F|E) \text{ provided } P(E) \neq 0$$
$$= P(F)\,P(E|F) \text{ provided } P(F) \neq 0.$$

If E, F, G are three events associated with a sample space, then

$$P(E \cap F \cap G) = P(E)\,P(F|E)\,P(G|(E \cap F))$$

Independent Events : Let E and F be two events, then if probability of occurrence one of them is not affected by the occurrence of the other, then E and F are said to be independent events, i.e.,

(a) $P(F|E) = P(F), \quad P(E) \neq 0$

or (b) $P(E|F) = P(E), \quad P(F) \neq 0$

or (c) $P(E \cap F) = P(E)\,P(F)$

Three events A, B, C are mutually independent if

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$
$$P(A \cap B) = P(A)\,P(B)$$
$$P(B \cap C) = P(B)\,P(C)$$

and $P(A \cap C) = P(A)\,P(C)$

Partition of a Sample Space : A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of a sample space S if

(a) $E_i \cap E_j = \phi$; $i \neq j$; $i, j = 1, 2, 3, ..., n$

(b) $E_1 \cup E_2 \cup E_3 ... \cup E_n = S$ and

(c) Each $E_i \neq \phi$ i.e. $P(E_i) > 0 \;\forall\; i = 1, 2, ..., n$

Theorem of Total Probability : Let $\{E_1, E_2, ..., E_n\}$ be a partition of the sample space S. Let A be the any event associated with S, then

$$P(A) = \sum_{j=1}^{n} P(E_j)\, P(A|E_j)$$

Baye's Theorem : If E_1, E_2, ..., E_n are mutually exclusive and exhaustive events associated with a sample space S, and A is any event associated with E_i's having non-zero probability, then

$$P(E_i|A) = \frac{P(A \mid E_i)P(E_i)}{\sum_{i=1}^{n} P(A \mid E_i)P(E_i)}$$

Illustration:

Evaluate $P(A \cup B)$ if $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$

Solution: $2P(A) = P(B) = \frac{5}{13}$

$\Rightarrow$ $P(A) = \frac{5}{26}$, $P(B) = \frac{5}{13}$

As $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$\Rightarrow \frac{2}{5} = \frac{P(A \cap B)}{(5/13)} \quad \Rightarrow \quad \frac{2}{\cancel{5}} \times \frac{\cancel{5}}{13} = P(A \cap B)$$

$$\Rightarrow \frac{2}{13} = P(A \cap B)$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{16}$$

Illustration:

Prove that if E and F are independent events, then the events E and F′ are also independent.

Solution: $P(E \cap F) = P(E)\,P(F)$ (given)

Consider, $P(E \cap F') = P(E) - P(E \cap F)$

$= P(E) - P(E)\,P(F)$

$= P(E)\,(1 - P(F))$

$P(E \cap F') = P(E) - P(F')$

So, E and F′ are also independent.

Illustration:

A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn. What is the probability that they both are diamonds.

Solution: Let E_1 = lost card is diamond

E_2 = lost card is non-diamond

A = 2 diamonds cards are drawn from the remaining cards

Using Theorem of total probability

$$P(A) = P(A|E_1)\,P(E_1) + P(A|E_2)\,P(E_2)$$

$$= \frac{12}{51} \times \frac{11}{50} \times \frac{\cancel{13}}{\cancel{52}_4} + \frac{12}{50} \times \frac{13}{51} \times \frac{\cancel{39}^3}{\cancel{52}_4}$$

$$= \frac{132}{10200} + \frac{468}{10200} = \frac{\cancel{600}}{\cancel{10200}_{17}} = \frac{1}{17}$$

ONE MARK QUESTIONS

1. The events E and F are independent. If P(E) = 0.3 and P(EUF) = 0.5, then P(E/F) – P(F/E) equals:

   (a) $\frac{1}{7}$ (b) $\frac{2}{7}$

   (c) $\frac{3}{35}$ (d) $\frac{1}{70}$

2. For two events A and B, if P(A) = 0.4, P(B) = 0.8, P(B/A) = 0.6, then P(AUB) is:

   (a) 0.24 (b) 0.3

   (c) 0.48 (d) 0.96

3. If A and B are two events such that P(A/B) = 2 × P(B/A) and P(A) + P(B) = $\frac{2}{3}$, then P(B) is equal to

   (a) $\frac{2}{9}$ (b) $\frac{7}{9}$

   (c) $\frac{7}{9}$ (d) $\frac{5}{9}$

4. Two events A and B will be independent, if;

   (a) A and B are mutually exclusive

   (b) P(A) = P(B)

   (c) P(A′B′) = [1 – P(A)] [1 – P(B)]

   (d) P(A) + P(B) = 1

5. If for any two events A and B, P(A) = $\frac{4}{5}$ and P(A $\cap$ B) = $\frac{7}{10}$, then P(B/A) is equal to

   (a) $\frac{1}{10}$ (b) $\frac{1}{8}$

   (c) $\frac{7}{8}$ (d) $\frac{17}{20}$

6. Five fair coins are tossed simultaneously. The probability of the events that atleast one head comes up is

   (a) $\frac{27}{32}$ (b) $\frac{5}{32}$

   (c) $\frac{31}{32}$ (d) $\frac{1}{32}$

7. Ashima can hit a target 2 out of 3 times. She tried to hit the target twice. The probability that she missed the target exactly one is

   (a) $\frac{2}{3}$ (b) $\frac{1}{3}$

   (c) $\frac{4}{9}$ (d) $\frac{1}{9}$

8. If sum of numbers obtained on throwing a pair of dice is 9, then the probability that number obtained on one of the dice is 4 is

(a) $\frac{1}{9}$ (b) $\frac{4}{9}$

(c) $\frac{1}{18}$ (d) $\frac{1}{2}$

9. X & Y are independent events such that $P(x \cap \bar{y}) = \frac{2}{5}$ and $P(X) = \frac{3}{5}$. Then P(Y) is equal to

(a) $\frac{2}{3}$ (b) $\frac{2}{5}$

(c) $\frac{1}{3}$ (d) $\frac{1}{5}$

10. If for two events A and B, P(A – B) = $\frac{1}{5}$ and P(A) = $\frac{3}{5}$, then $P\left(\frac{B}{A}\right)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{3}{5}$

(c) $\frac{2}{5}$ (d) $\frac{2}{3}$

11. If A and B are two events such that P(A) > 0 and P(B) ≠ 1, then $P(\bar{A} \mid \bar{B}) =$

(a) $1 - P(A/B)$ (b) $1 - P(A/\bar{B})$

(c) $\frac{1 - P(A \cup B)}{P(B)}$ (d) $\frac{P(\bar{A})}{P(B)}$

12. A and B are events such that P(A/B) = P(B/A) then

(a) $A \subset B$ (b) B = A

(c) $A \cap B = \phi$ (d) P(A) = P(B)

13. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2 respectively. The second plane will bomb only if the first misses the target. The probability that the target is hit by the II plane is

(a) 0.2 (b) 0.7

(c) 0.06 (d) 0.14

14. P(E $\cap$ F) is equal to

(a) P(E) P(F/E) (b) P(F).P(E/F)

(c) Both (a) & (b) (d) None of these

15. Two dice are thrown. If it is known that the sum of the numbers on the dice is less than 6, the probability of getting a sum 3 is

(a) $\frac{1}{8}$ (b) $\frac{2}{5}$

(c) $\frac{1}{5}$ (d) $\frac{5}{18}$

In following questions Q16 to Q20, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A)

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A)

(c) (A) is true and (R) is false

(d) (A) is false but (R) is true

16. Assertion (A) : Let A and B be two independent events. The $P(A \cap B) = P(A) + P(B)$

Reason (R) : Three events A, B and C are said to be independent if

$P(A \cap B \cap C) = P(A).P(B).P(C)$

17. Assertion (A) : Two coins are tossed simultaneously. The probabiltiy ofgetting two heads, if it is known that atleast one head comes up is $\frac{1}{3}$.

Reason (R) : Let E and F be two events with a random experiment, then

$P(F/E) = \frac{P(F \cap E)}{P(E)}$

18. Assertion (A) : Bag P contains 6 Red and 4 Blue balls and Bag Q contains 5 red and 6 Blue Balls. A ball is transferred from Bag P to bag Q and then a ball is drawn from Bag Q. The probability that the ball drawn from bag Q is blue is $\frac{8}{15}$.

Reason (R) : According to the law of total probability

$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$ where E_1 and E_2 partitions the sample space S and A is any event connected with E_1 and E_2.

TWO MARKS QUESTIONS

1. A and B are two events such that $P(A) \neq 0$, then find $P(B|A)$ if (*i*) A is a subset of B (*ii*) $A \cap B = \phi$.

2. Out of 30 consecutive integers two are chosen at random. Find the probability so that their sum is odd.

3. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. Find the probability that the eldest child is a girl given that the family has atleast one girl.

4. If A and B are such that $P(A \cup B) = \frac{5}{9}$ and $P(\bar{A} \cup \bar{B}) = \frac{2}{3}$, then find $P(\bar{A}) + P(\bar{B})$.

5. Prove that if A and B are independent events, then A and B' are also independent events.

6. If A and B are two independent events such that $P(A) = 0.3$, $P(A \cup B) = 0.5$, then find $P(A|B) - P(B|A)$

7. Three faces of an ordinary dice are yellow, two faces are red and one face is blue. The dice is rolled 3 times. Find the probability that yellow, red and blue face appear in the first, second and third throw respectively.

8. Find the probability that a leap year will have 53 Fridays or 53 Saturdays.

9. A person writes 4 letters and addresses on 4 envelopes. If the letters are placed in the envelopes at random, then what is the probability that all the letters are not placed in the right envelopes.

10. In a class XII of a school, 40% of students study Mathematics, 30% of the students study Biology and 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, then find the probability that he will be studying Mathematics or Biology.

THREE MARKS QUESTIONS

Q.1. A problem in mathematics is given to three students whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. What is the probability that the problem is solved ?

Q.2. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$ then find $P(A)$ and $P(B)$.

Q.3. Amit and Nisha appear for an interview for two vacancies in a company. The probability of Amit's selection is 1/5 and that of Nisha's selections is 1/6. What is the probability that

(i) both of them are selected?

(ii) only one of them is selected?

(iii) none of them is selected?

Q.4. Suppose that 10% of men and 5% of women have grey hair. A grey haired person is selected at random. What is the probability that the selected person is male assuming that there are 60% males and 40% females ?

Q.5. Two dice are thrown once. Find the probability of getting an even number on the first die or a total of 8.

Q.6. Two aeroplanes X and Y bomb a target in succession. There probabilities to hit correctly are 0.3 and 0.2 respectively. The second plane will bomb only if first miss the target. Find the probability that target is hit by Y plane.

Q.7. A and B are independent events such that $P(A \cap \bar{B}) = \frac{1}{4}$ and $P(\bar{A} \cap B) = \frac{1}{6}$. Find P(A) and P(B).

Q.8. There are two coins. One of them is a biased coin such that

P(Head) : P(tail) is 1 : 3 and the other is a fair coin. A coin is selected at random and tossed once. If the coin showed head, then find the probability that it is a biased coin.

Q.9. A fair coin and an unbiased die are tossed. Let A be the event "Head appears on the coin" and B' be the event, "3 comes on the die". Find whether A and B are independent or not.

FIVE MARKS QUESTIONS

Q.1. By examining the chest X-ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of a healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB ?

Q.2. Three persons *A*, *B* and *C* apply for a job of Manager in a private company. Chances of their selection (*A*, *B* and *C*) are in the ratio 1 : 2 : 4. The probabilities that *A, B* and *C* can introduce charges to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the change doesn't take place, find the probability that it is due to the appointment of *C*.

Q.3. A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope, just two consecutive letters TA are visible. What is the probability that the letter came from TATANAGAR.

Q.4. Three critics review a book. Odds in favour of the book are 5 : 2, 4 : 3 and 3 : 4 respectively for the three critics. Find the probability that the majority are in favour of the book.

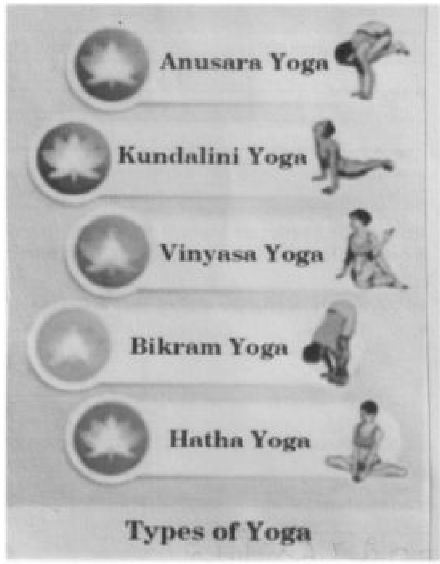
Q.5. An urn contains five balls. Two balls are drawn and are found to be white. What is the probability that all the balls are white?

Q.6. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn at random and are found to be both clubs. Find the possibility of the lost card being of club.

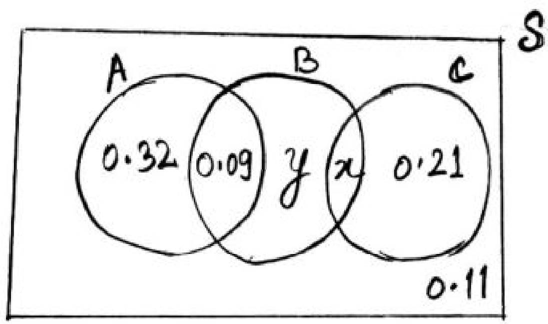
Q.7. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II at random. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

CASE STUDY QUESTIONS

Q.1. There are different types of Yoga which involve the usage of different poses of Yoga Asanas, Meditation and Pranayam as shown in the figure below:



The Venn Diagram below represents the probabilities of three different types of Yoga, A, B and C performed by the people of a society. Further it is given that probability of a member performing type C Yoga is 0.44.



On the basis of the above information, answer the following questions:

(i) Find the value of x.

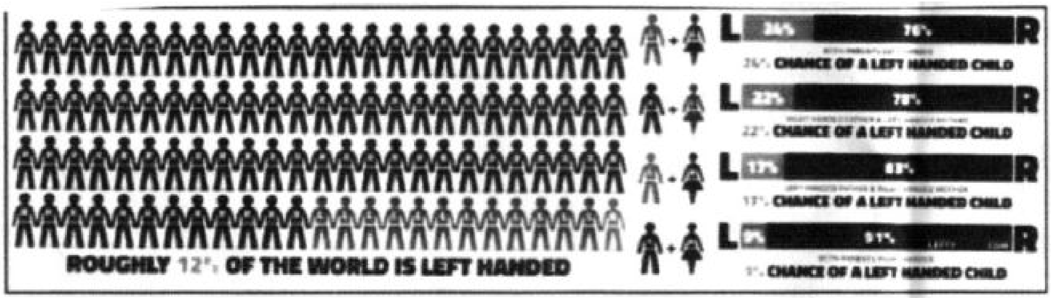
(ii) Find the value of y.

(iii) (a) Find $P\left(\frac{C}{B}\right)$

OR

(iii) (b) Find the probability that a randomly selected person of the society does Yoga of type A or B but not C.

Q.2. Recent studies suggest that roughly 12% of the world population is left handed.



Depending upon the parents, the chances of havig a left handed child are as follows:

A : When both father and mother are left handed

Chances of left handed child is 24%

B : When father is right handed and mother is left handed:

Chances of left handed child is 22%

C : When father is left handed and mother is right handed:

Chances of left handed child is 17%

D : When both father and mother are right handed:

Chances of left handed child is 9%

Assuming that $P(A) = P(B) = P(C) = P(D) = \frac{1}{4}$ and L denotes the event that child is left handed.

Based on the above information, answer the following questions:

(i) Find P(L/C)

(ii) Find $P(\bar{L}/A)$

(iii) (a) Find P(A/L)

OR

(b) Find the probability that a randomly selected child is left handed given that exactly one of the parents is left handed.

Q.3. In a birthday party, a magician was being invited by a parent and he had 3 bags that contain number of red and white balls as follows:

Bag 1 contains : 3 red balls, Bag 2 contains : 2 white balls and 1 Red ball

Bag 3 contains : 3 white balls

The probability that the bag *i* will be chosen by the magician and a ball is selected from it is $\frac{i}{6}, i = 1, 2, 3$.

Based on the above information, answer the following questions.

(a) What is the probability that a red ball is selected by the magician?

(b) What is the probability that a white ball is selected by the magician?

(c) Given that the magician selects the white ball, what is the probability that the ball was from Bag 2.

Q.4. In an office three employees Vinay, Sonia and Iqbal process incoming copies of a certain form. Vinay process 50% of the forms. Sonia processes 20% and Iqbal the remaining 30% of the forms. Vinay has an error rate of 0.06, Sonia has an error rate of 0.04 and Iqbal has an error rate of 0.03.

Based on the above information answer the following :

(*i*) Find the conditional probability that an error is committed in processing given that Sonia processed the form?

(*ii*) What is probability that Sonia processed the form and committed an error?

(*iii*) What is total probability of committing an error in processing the form?

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If A and B are independent events such that P(A) = 0.4, P(B) = x and P(A $\cup$ B) = 0.5, then x = ?

(a) $\frac{4}{5}$ (b) 0.1

(c) $\frac{1}{6}$ (d) None of these

2. If P(A) = 0.8, P(B) = 0.5 and P(B/A) = 0.4, then P(A/B) is

(a) 0.32 (b) 0.64

(c) 0.16 (d) 0.25

3. A couple has two children. What is the probability that both are boys if it is known that one of them is a boy?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{3}{4}$ (d) $\frac{1}{4}$

4. If two events are independent, then

(a) they must be mutually exclusive

(b) the sum of their probabilities must be equal to 1

(c) (a) and (b) both are correct

(d) none of the above is correct

SELF ASSESSMENT-2

EAH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Two cards are drawn from a well shuffled deck of 52 playing cards with replacement. The probability that both cards are queens is

(a) $\frac{1}{13}\times\frac{1}{13}$ (b) $\frac{1}{13}+\frac{1}{13}$

(c) $\frac{1}{13}\times\frac{1}{17}$ (d) $\frac{1}{13}\times\frac{4}{51}$

2. Three persons A, B and C fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is

(a) 0.024 (b) 0.188

(c) 0.336 (d) 0.452

3. If $4P(A) = 6P(B) = 10P(A \cap B) = 1$, then $P(B/A) = ?$

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$

(c) $\frac{7}{10}$ (d) $\frac{19}{60}$

4. A letter is known to have come either from LONDON or CLIFTON; on the postmark only the two consecutive letters ON are legible. The probability that it came from LONDON is

(a) $\frac{5}{17}$ (b) $\frac{12}{17}$

(c) $\frac{17}{30}$ (d) $\frac{3}{5}$

ANSWER

One Mark Questions

1. (d) $\frac{1}{70}$	2. (d) 0.96	3. (a) $\frac{2}{9}$	4. (c)
5. (c) $\frac{7}{8}$	6. (c) $\frac{31}{32}$	7. (c) $\frac{4}{9}$	8. (d) $\frac{1}{2}$
9. (c) $\frac{1}{3}$	10. (d) $\frac{2}{3}$	11. (b) $1 - P(A/\overline{B})$	12. (d) P(A) = P(B)
13. (d) 0.14	14. (c) Both (a) & (b)	15. (c) $\frac{1}{5}$	
16. (d)	17. (a)	18. (a)	

Two Marks Questions

1. (*i*) 1 (*ii*) 0
2. $\frac{15}{29}$
3. $\frac{4}{7}$
4. $\frac{10}{9}$
6. $\frac{1}{70}$
7. $\frac{1}{36}$
8. $\frac{3}{7}$
9. $\frac{23}{24}$
10. 0.6

Three Marks Questions

1. $\frac{3}{4}$
2. $P(A) = \frac{1}{5}$ and $P(B) = \frac{1}{6}$ or $P(A) = \frac{5}{6}$ and $P(B) = \frac{4}{5}$
3. (i) $\frac{1}{30}$ (ii) $\frac{3}{10}$ (iii) $\frac{2}{3}$
4. $\frac{3}{4}$
5. $\frac{5}{9}$
6. $\frac{7}{50}$
7. $\frac{1}{3}$ and $\frac{1}{4}$, $\frac{3}{4}$ and $\frac{2}{3}$
8. $\frac{1}{3}$
9. Yes, A and B are independent.

Five Marks Questions

1. $\frac{110}{221}$
2. $\frac{7}{10}$
3. $\frac{7}{11}$

4. $\frac{209}{343}$ 5. $\frac{1}{2}$.

6. $\frac{11}{50}$ 7. $\frac{16}{31}$

CASE STUDY QUESTIONS

1. (i) x = 0.23 (ii) y = 0.04 (iii) (a) $\frac{23}{36}$ or (b) 0.46

2. (i) P(L/C) = 0.17, (ii) $P(\bar{L}/A) = 0.76$ (iii) (a) $P(A/L) = \frac{1}{3}$ or (*b*) 0.39

3. (*a*) $\frac{5}{18}$ (*b*) $\frac{13}{18}$ (*c*) $\frac{4}{13}$

4. (*i*) 0.04 (*ii*) 0.008 (*iii*) 0.047

SELF ASSESSMENT-1

1. (c) 2. (b) 3. (a) 4. (d)

SELF ASSESSMENT-2

1. (a) 2. (b) 3. (a) 4. (b)

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